

# MULTIPLIERS BETWEEN MODEL SPACES

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**ABSTRACT.** In this paper we characterize the multipliers from one model space (of the disk) to another. Our characterization involves kernels of Toeplitz operators and Carleson measures. We illustrate this characterization in different situations and in large classes of examples. As it turns out, under certain circumstances, every multiplier between the two model spaces is a bounded function. However, this is not always the case. In the case of onto multipliers, this answers a question posed by Crofoot [14]. When considering model spaces of the upper-half plane, we will discuss in some detail when the associated inner function is a meromorphic inner function. This connects to de Branges spaces of entire functions which are closely related to different important problems in complex analysis (e.g., zero distribution, differential equations, and completeness problems). When the derivative of the associated inner function is bounded, we show that the set of multipliers contains the kernel of an associated Toeplitz operator.

## 1. INTRODUCTION

For two Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  of analytic functions on the open unit disk  $\mathbb{D}$ , what are the multipliers from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ ? By the term *multiplier*, we mean a  $\varphi \in \mathcal{O}(\mathbb{D})$  (the analytic functions on  $\mathbb{D}$ ) for which  $\varphi\mathcal{H}_1 \subseteq \mathcal{H}_2$ . The set of all multipliers from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  will be denoted by  $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ . The set of bounded multipliers, i.e.,  $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \cap H^\infty$ , where  $H^\infty$  denotes the bounded analytic functions on  $\mathbb{D}$ , will be denoted by  $\mathcal{M}_b(\mathcal{H}_1, \mathcal{H}_2)$ . Note that  $\mathcal{M}_b(\mathcal{H}_1, \mathcal{H}_2) \subseteq \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$  but there are cases, as we will see below, when this inclusion is proper.

For example, when  $\mathcal{H}_1 = \mathcal{H}_2 = H^2$ , the classical Hardy space [17, 29], it is well known (see also Proposition 3.1 below) that  $\mathcal{M}(H^2, H^2) = H^\infty$ . The same is true, with nearly the same proof, when  $\mathcal{H}_1 = \mathcal{H}_2 = A^2$ , the Bergman space [18]. When  $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{D}$ , the Dirichlet

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space, the situation is more delicate in that  $\mathcal{M}(\mathcal{D}, \mathcal{D}) \subsetneq H^\infty$  [23, 50]. There are results from [24, 50] which examine the multiplier spaces  $\mathcal{M}(H^2, A^2)$ ,  $\mathcal{M}(\mathcal{D}, H^2)$ , and  $\mathcal{M}(\mathcal{D}, A^2)$ . However, since the zeros sequences for  $\mathcal{D}$ ,  $H^2$ , and  $A^2$  are different, meaning, for example, there are  $A^2$  functions whose zeros are not those of any (non-zero)  $H^2$  function [18, p. 94], one concludes that  $\mathcal{M}(A^2, H^2) = \mathcal{M}(H^2, \mathcal{D}) = \mathcal{M}(A^2, \mathcal{D}) = \{0\}$ .

In this paper, we explore, for a pair of inner functions  $u$  and  $v$ , the multipliers  $\mathcal{M}(\mathcal{K}_u, \mathcal{K}_v)$ , abbreviated  $\mathcal{M}(u, v)$ . Here, for an inner function  $\Theta$  (a bounded analytic function on  $\mathbb{D}$  with unimodular values almost everywhere on  $\mathbb{T} = \partial\mathbb{D}$ ),

$$\mathcal{K}_\Theta := H^2 \cap (\Theta H^2)^\perp$$

will denote the *model space* corresponding to  $\Theta$  [27, 40].

One motivation for this paper comes from the work of Crofoot [14] who considered a more restricted version of our model space multiplier problem ( $\varphi\mathcal{K}_u \subseteq \mathcal{K}_v$ ), namely the class of  $\varphi \in \mathcal{O}(\mathbb{D})$  for which  $\varphi\mathcal{K}_u = \mathcal{K}_v$ , in other words, the multipliers from  $\mathcal{K}_u$  onto  $\mathcal{K}_v$  (see also [8, Def. 3.7] where the authors are interested in the onto multipliers which are bounded and boundedly invertible). As it turns out, these onto multipliers are unique up to multiplicative constants and are outer functions. Furthermore, Crofoot [14, Thm. 17] characterized the onto multipliers in terms of their arguments and certain Carleson measure conditions. More precisely,  $\varphi\mathcal{K}_u = \mathcal{K}_v$  if and only if  $(v\bar{\varphi})/(u\varphi)$  is a constant function on  $\mathbb{T}$  and

$$(1.1) \quad \varphi\mathcal{K}_u \subseteq H^2 \text{ and } \frac{1}{\varphi}\mathcal{K}_v \subseteq H^2.$$

We will present a new version of this result in Section 10. Crofoot also showed that for every  $a \in \mathbb{D}$ ,

$$(1.2) \quad \frac{1}{1 - \bar{a}u}\mathcal{K}_u = \mathcal{K}_{u_a},$$

where  $u_a$  is the inner function defined by

$$(1.3) \quad u_a := \frac{u - a}{1 - \bar{a}u},$$

i.e., a Frostman shift of  $u$ . Notice how the (onto) multiplier  $(1 - \bar{a}u)^{-1}$  from (1.2) is an outer function.

An issue left unresolved in Crofoot's paper [14, p. 244] was whether or not this (onto) multiplier  $\varphi$  must always be a bounded function. In

Theorem 8.4 of this paper, we use de Branges spaces of entire functions [15] to resolve this issue with the following:

**Theorem.** *There are two inner functions  $u$  and  $v$  and an unbounded  $\varphi \in \mathcal{O}(\mathbb{D})$  such that  $\varphi\mathcal{K}_u = \mathcal{K}_v$ .*

Notice that since the onto multipliers are unique up to multiplicative constants, the theorem above produces a non-trivial multiplier space that, except for the zero multiplier, contains no bounded functions.

The above discussion becomes quite different if we relax the (onto) multiplier condition  $\varphi\mathcal{K}_u = \mathcal{K}_v$  to just

$$\varphi\mathcal{K}_u \subseteq \mathcal{K}_v.$$

For one, these (into but not necessarily onto) multipliers need not be outer functions. To see this, let  $I$  and  $J$  be two non-constant inner functions. The well-known orthogonal decomposition  $\mathcal{K}_{IJ} = \mathcal{K}_J \oplus J\mathcal{K}_I$  [27, Ch. 5] shows that  $J \in \mathcal{M}(I, IJ)$ . Secondly, unlike the onto multipliers, the into multipliers need not be unique (see Theorem 4.10 below). As we have seen, they also need not be bounded. In Section 8 we will give several other examples of this.

A more recent source of inspiration for this paper stems from [28] and a recent preprint [51] which examined various pre-orders on the set of partial isometries [28] and contractions [51] on Hilbert spaces and their relationship to their associated Livšic characteristic functions. It turns out, for example, that when the Livšic characteristic functions  $u$  and  $v$  for two partial isometries  $A$  and  $B$  are inner, the issue of whether or not  $A$  is “less than”  $B$  can be rephrased as to whether or not  $\mathcal{M}(u, v) \neq \{0\}$ .

To state another of the main results of this paper, the description of  $\mathcal{M}(u, v)$ , and to further connect our work with some well-studied problems in analysis, we recall a few definitions that will be explained in later sections. The model spaces  $\mathcal{K}_u$  are the generic invariant subspaces for the backward shift operator  $S^*$  on  $H^2$  and are singly generated by the vector  $S^*u$  (see (2.13) below). For  $\varphi \in H^2$ , we say that the measure  $|\varphi|^2 dm$ , where  $m$  is normalized Lebesgue measure on the unit circle  $\mathbb{T}$ , is a *Carleson measure* for  $\mathcal{K}_u$  if

$$(1.4) \quad \int_{\mathbb{T}} |f|^2 |\varphi|^2 dm \lesssim \int_{\mathbb{T}} |f|^2 dm, \quad f \in \mathcal{K}_u.$$

Certainly any Carleson measure for  $H^2$  [29] is also a Carleson measure for  $\mathcal{K}_u$  but, for an inner function  $u$ , there are Carleson measures for

$\mathcal{K}_u$  which need not be Carleson measures for  $H^2$  (the associated Clark measure is such an example – see below). Such measures and their generalizations have been discussed quite thoroughly in the papers [2, 3, 4, 5, 11, 12, 53] (and recently characterized in [36]) and we will use some of these results in our analysis. For now, notice from (1.4) that  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$  precisely when  $\varphi \in \mathcal{M}(\mathcal{K}_u, H^2)$  and how these measures appeared in our previous discussion of Crofoot's results (see (1.1)). Our main theorem will also involve the kernels of Toeplitz operators  $T_\psi$  on  $H^2$  which are well explored territory and we will make use of results from [6, 21, 31, 32, 33, 39, 47] (see also [30] for a survey).

We first note that  $\mathcal{M}(u, v) \subseteq H^2$  (see 3.7 below). Our description of  $\mathcal{M}(u, v)$  is as follows:.

**Theorem.** *For inner functions  $u$  and  $v$  and  $\varphi \in H^2$ , the following are equivalent:*

- (i)  $\varphi \in \mathcal{M}(u, v)$ ;
- (ii)  $\varphi S^*u \in \mathcal{K}_v$  and  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ .
- (iii)  $\varphi \in \text{Ker } T_{\overline{v}u}$  and  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ .

Furthermore, the following are equivalent:

- (iv)  $\varphi \in \mathcal{M}_b(u, v)$ ;
- (v)  $\varphi S^*u \in \mathcal{K}_v \cap H^\infty$ .
- (vi)  $\varphi \in \text{Ker } T_{\overline{v}u} \cap H^\infty$ .

So, in order for  $\varphi$  to be a multiplier from  $\mathcal{K}_u$  into  $\mathcal{K}_v$ , it must first be a multiplier from  $\mathcal{K}_u$  to  $H^2$  (an important necessary condition and described by the Carleson condition (1.4)) and then  $\varphi$  must simply multiply the test function  $S^*u \in \mathcal{K}_u$  into  $\mathcal{K}_v$ .

As one can see from statements (iii) and (vi) above, our main theorem leads to a discussion of  $\text{Ker } T_{\overline{I}J}$  for inner  $I$  and  $J$ . This turns out to be an important and well-studied problem which makes connections to several topics in analysis [21, 35, 39, 41].

Our paper is structured as follows. After setting our notation and reminding the reader of some elementary facts about model spaces in Section 2, we proceed to Section 3 where we both review some of the results from Crofoot's paper [14] as well as establish some elementary facts about  $\mathcal{M}(u, v)$  (e.g.,  $\mathcal{M}(u, u) = \mathbb{C}$ ;  $\mathcal{M}(u, v) \neq \{0\}$  implies the boundary spectrum of  $u$  is contained in the boundary spectrum

of  $v$ ;  $u$  a finite Blaschke product and  $v$  any inner function of infinite degree implies  $\mathcal{M}(u, v) \neq \{0\}$ ). Section 4 contains our main result (described above) along with a description of  $\mathcal{M}(u, uI)$  and  $\mathcal{M}(u, v)$  (where  $u$  and  $v$  are finite Blaschke products) as well as examples of when  $\mathcal{M}(u, v) \subsetneq \text{Ker } T_{\overline{v}u}$ . In Section 5 we use a maximum principle of Cohn [13] to establish a sufficient condition as to when  $\mathcal{M}(u, v) \subseteq H^\infty$  via the sub-level sets of  $u$  and  $v$ . We also give an alternative description of  $\mathcal{M}(u, v)$ , involving a more tractable testing condition, when  $u$  satisfies the so-called connected level set condition. Section 6 contains some examples of our main theorem involving the Frostman shifts of inner functions and the Crofoot transform. A version of the multiplier problem for model spaces for the upper half plane is explored in Section 7. This upper-half plane setting has some interesting features. When the derivative of the associated inner function is bounded, the Carleson measure condition becomes rather easy to check. In particular, the multipliers contained in the Hardy space of the upper half plane  $\mathcal{H}^2$  are exactly given by an associated Toeplitz kernel. The upper-half plane setting also lays the groundwork for our resolution of the Crofoot question concerning the existence of unbounded onto multipliers discussed in Section 8. Moreover, we will show that although the multipliers preserve the spectra of the two inner functions (see Proposition 3.30), they do not, in general, preserve the well-known Ahern-Clark condition (see Proposition 7.16). Section 8 contains several examples of inner functions  $u$  and  $v$  for which  $\mathcal{M}(u, v)$  contains unbounded functions, one uses analytic continuation, another using Bourgain factorization, and still another, yielding a negative answer to Crofoot's question, using de Branges spaces of entire functions. We bring in the topic of Clark measures and rephrase some of our multiplier results in this setting in Section 9, which also allows us to create further classes of interesting non-trivial classes of multiplier spaces  $\mathcal{M}(u, v)$ . We give several reformulations of Crofoot's description of the onto multipliers in Section 10 while in Section 11 we state some generalizations to the  $L^p$  setting as well as pose some topics for further discussion.

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## 2. NOTATION AND BASIC FACTS

We assume the reader is familiar with Hardy spaces and so this section is to set our notation and remind the reader of some of the basics of

model spaces. Sources for this Hardy space material include [17, 29] while sources for model spaces include [27, 40]. Throughout this paper,  $\mathbb{D}$  will be the open unit disk,  $\mathbb{T}$  the unit circle,  $m$  normalized Lebesgue measure on  $\mathbb{T}$ , and  $L^2$  the standard Lebesgue space  $L^2 := L^2(\mathbb{T}, m)$  with norm

$$\|f\| := \left( \int_{\mathbb{T}} |f|^2 dm \right)^{1/2}$$

and corresponding  $L^2$  inner product  $\langle \cdot, \cdot \rangle$ .

**Hardy space of the disk.** Recall the “vanishing negative Fourier coefficients” definition of the *Hardy space*

$$(2.1) \quad H^2 := \{f \in L^2 : \langle f, \zeta^n \rangle = 0 \ \forall n < 0\}.$$

The Hardy space is a closed subspace of  $L^2$  and corresponds to a Hilbert space of analytic functions on  $\mathbb{D}$  via “bounded integral means”: If  $f \in \mathcal{O}(\mathbb{D})$ , then

$$(2.2) \quad f \in H^2 \iff \sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\xi)|^2 dm(\xi) < \infty.$$

By means of radial limit functions, the two characterizations of  $H^2$  from (2.1) and (2.2) coincide.

Every  $f \in H^2$  can be factored as

$$(2.3) \quad f = uF,$$

where  $u$  is an inner function and  $F \in H^2$  and is outer. Every inner function  $u$  can be factored as

$$(2.4) \quad u = B_{\Lambda} s_{\mu},$$

where  $B_{\Lambda}$  is the Blaschke factor with zero sequence  $\Lambda \subseteq \mathbb{D}$  (repeated according to multiplicity) and  $s_{\mu}$  is the singular inner factor with positive singular measure  $\mu$  on  $\mathbb{T}$ . The factorizations (2.3) and (2.4) are unique up to a multiplicative unimodular constant.

The Hardy space is also a reproducing kernel Hilbert space with Cauchy kernel

$$(2.5) \quad k_{\lambda}(z) := \frac{1}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D},$$

in that

$$f(\lambda) = \langle f, k_{\lambda} \rangle, \quad \lambda \in \mathbb{D}, f \in H^2.$$

**Model spaces of the disk.** For an inner function  $u$  define the *model space*

$$\mathcal{K}_u := H^2 \cap (uH^2)^\perp.$$

These are often called the “pseudocontinuable functions” and have an equivalent description as “matching boundary values” with a meromorphic function on the exterior disk [16, 44]. If  $S$  is the forward shift operator  $Sf := zf$  on  $H^2$ , then, by Beurling’s theorem [17],  $uH^2$ , where  $u$  is a non-constant inner function, are all of the non-trivial  $S$ -invariant subspaces of  $H^2$ . One can show that

$$S^*f = \frac{f - f(0)}{z},$$

and thus  $\mathcal{K}_u$ , where  $u$  is a non-constant inner function, comprise all of the non-trivial  $S^*$ -invariant subspaces properly contained in  $H^2$ .

Here are some well-known facts about model spaces we will use in this paper. Two references for model spaces, containing the proofs of the facts mentioned below, are [27, 40]. The first simple but useful fact is

$$(2.6) \quad u(0) = 0 \iff 1 \in \mathcal{K}_u.$$

In terms of almost everywhere defined boundary functions on  $\mathbb{T}$ ,  $\mathcal{K}_u$  can be written alternatively as

$$(2.7) \quad \mathcal{K}_u = H^2 \cap \overline{uzH^2}.$$

Model spaces are reproducing kernel Hilbert spaces with reproducing kernel

$$(2.8) \quad k_\lambda^u(z) := \frac{1 - \overline{u(\lambda)}u(z)}{1 - \bar{\lambda}z},$$

meaning that  $k_\lambda^u \in \mathcal{K}_u$  for all  $\lambda \in \mathbb{D}$  and

$$f(\lambda) = \langle f, k_\lambda^u \rangle, \quad f \in \mathcal{K}_u.$$

Furthermore, each  $k_\lambda^u$  belongs to  $H^\infty$ , is outer, is invertible in  $H^\infty$ , and satisfies

$$(2.9) \quad \bigvee \{k_\lambda^u : \lambda \in \mathbb{D}\} = \mathcal{K}_u,$$

where  $\bigvee$  denotes the closed linear span in the  $H^2$  norm.

We also have the following inclusion result for model spaces

$$(2.10) \quad \mathcal{K}_u \subseteq \mathcal{K}_v \iff \frac{v}{u} \in H^\infty.$$

When  $v/u \in H^\infty$  one often says that  $u$  *divides*  $v$ , which is equivalent to saying that  $v = uI$  for some inner function  $I$ . As a corollary, observe that

$$(2.11) \quad \mathcal{K}_u = \mathcal{K}_v \iff u = \xi v,$$

for some unimodular constant  $\xi$ . When  $v = uI$  we have the orthogonal decomposition

$$(2.12) \quad \mathcal{K}_v = \mathcal{K}_u \oplus u\mathcal{K}_I = \mathcal{K}_I \oplus I\mathcal{K}_u.$$

The model spaces are also singly generated by  $S^*u$  in that

$$(2.13) \quad \bigvee \{S^{*n}u : n \geq 1\} = \mathcal{K}_u.$$

We already mentioned that model spaces have pseudo-continuation properties. In certain circumstances, they also have analytic continuation properties. Indeed, if

$$(2.14) \quad \sigma(u) := \left\{ \zeta \in \mathbb{T} : \lim_{z \rightarrow \zeta} |u(z)| = 0 \right\}$$

is the *boundary spectrum* of  $u$ , then, assuming that  $\sigma(u) \neq \mathbb{T}$ , there is a two dimensional open neighborhood  $\Omega$  of  $\mathbb{T} \setminus \sigma(u)$  such that

$$(2.15) \quad \text{every } f \in \mathcal{K}_u \text{ has an analytic continuation to } \Omega.$$

Furthermore, if  $\zeta \in \sigma(u)$ , then  $S^*u \in \mathcal{K}_u$  does not have an analytic continuation to an open neighborhood of  $\zeta$ . Thus  $\mathbb{T} \setminus \sigma(u)$  is the maximal set of points for which every function in  $\mathcal{K}_u$  has an analytic continuation. The set  $\mathbb{T} \setminus \sigma(u)$  is sometimes called the *regular* points of  $u$ . If  $u = B_\Lambda s_\mu$  (Blaschke factor  $B_\Lambda$  and singular inner factor  $s_\mu$ ) then

$$(2.16) \quad \sigma(u) = \mathbb{T} \cap (\Lambda^- \cup \text{supt}(\mu)),$$

where  $\Lambda^-$  is the closure of  $\Lambda$  and  $\text{supt}(\mu)$  is the support of  $\mu$ .

There is also a more subtle result of Ahern and Clark [1] which discusses when every function in a model space has a radial (even a non-tangential) limit at  $\zeta \in \mathbb{T}$  even when  $\zeta \in \sigma(u)$ . The result says that for an inner function  $u$  with zero set  $\{a_n\}_{n \geq 1}$  (repeated according to multiplicity) and associated singular measure  $\mu$  from (2.4),

$$(2.17) \quad \begin{aligned} & \text{every } f \in \mathcal{K}_u \text{ has a non-tangential limit at } \zeta \\ & \iff \sum_{n \geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} + \int_{\mathbb{T}} \frac{1}{|\zeta - \xi|^2} d\mu(\xi) < \infty \\ & \iff \lim_{z \rightarrow \zeta} \frac{1 - |u(z)|}{1 - |z|} < \infty. \end{aligned}$$



The last equivalent condition above says that  $u$  has a finite *angular derivative* at  $\zeta$  and the point  $\zeta$  is called an *Ahern-Clark point* for  $\mathcal{K}_u$ .

There is the following characterization of when  $\mathcal{K}_u$  is finite dimensional, namely,

$$(2.18) \quad \begin{aligned} & \dim(\mathcal{K}_u) < \infty \\ & \iff u \text{ is a finite Blaschke product} \\ & \iff \sigma(u) = \emptyset. \end{aligned}$$

If the finite Blaschke product  $u$  has  $n$  zeros  $\{\lambda_1, \dots, \lambda_n\}$  (repeated according to their multiplicity), then

$$(2.19) \quad \mathcal{K}_u = \left\{ \frac{p(z)}{\prod_{j=1}^n (1 - \bar{\lambda}_j z)} : p \in \mathcal{P}_{n-1} \right\},$$

where  $\mathcal{P}_{n-1}$  are the (analytic) polynomials of degree at most  $n-1$ . In particular, for  $N \in \mathbb{N}$ ,  $\mathcal{K}_{z^N} = \mathcal{P}_{N-1}$ .

**Clark measures.** Given an inner function  $u$ , we can associate (via Herglotz's theorem on positive harmonic functions on  $\mathbb{D}$  [17, p. 3]) a unique positive finite measure  $\sigma_u$  on  $\mathbb{T}$  such that

$$(2.20) \quad \frac{1 - |u(z)|^2}{|1 - u(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \xi|^2} d\sigma_u(\xi), \quad z \in \mathbb{D}.$$

Moreover, using the fact that  $u$  is inner, it is easy to check that  $\sigma_u \perp m$ . Furthermore,  $u(0) = 0$  if and only if  $\sigma_u$  is a probability measure. This process can be reversed in that given any positive singular measure  $\nu$  on  $\mathbb{T}$ , there is an inner function  $u_\nu$  such that

$$(2.21) \quad \frac{1 - |u_\nu(z)|^2}{|1 - u_\nu(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \xi|^2} d\nu(\xi), \quad z \in \mathbb{D}.$$

A well-known theorem of Clark [10] says that the linear transformation  $V_u$  defined by

$$V_u(g) := (1 - u)C_{\sigma_u}(g), \quad g \in L^2(\sigma_u),$$

where

$$(C_{\sigma_u}g)(z) = \int_{\mathbb{T}} \frac{g(\xi)}{1 - z\bar{\xi}} d\sigma_u(\xi), \quad z \in \mathbb{D},$$

is the Cauchy transform associated with  $\sigma_u$ , is a unitary transformation from  $L^2(\sigma_u)$  onto  $\mathcal{K}_u$ . The measure  $\sigma_u$  is called the *Clark measure* corresponding to  $u$ . If  $k_\lambda$  is the standard Cauchy kernel from (2.5) and  $k_\lambda^u$  is the reproducing kernel for  $\mathcal{K}_u$  from (2.8) then

$$(2.22) \quad V_u k_\lambda = (1 - \overline{u(\lambda)})^{-1} k_\lambda^u, \quad \lambda \in \mathbb{D}.$$

Furthermore, a deep result of Poltoratski [43] says that every  $f \in \mathcal{K}_u$  has radial limits  $\sigma_u$ -almost everywhere and

$$(2.23) \quad V_u^{-1}(f) = f$$

on the carrier of  $\sigma_u$ . See [9, 42] for more on Clark measures.

This notion extends in several directions (and will be used from time to time in this paper). For one, there is actually a family of Clark measures  $\{\sigma_u^\alpha : \alpha \in \mathbb{T}\}$  associated with the inner function  $u$  by

$$\frac{1 - |u(z)|^2}{|\alpha - u(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\sigma_u^\alpha(\xi), \quad z \in \mathbb{D}.$$

Since  $u$  is an inner function, these measures  $\sigma_u^\alpha$  will all be singular with respect to Lebesgue measure.

Secondly, when  $u \in H^\infty$  and  $\|u\|_\infty \leq 1$ , one can still create a family of measures  $\{\sigma_u^\alpha : \alpha \in \mathbb{T}\}$ , called the Aleksandrov-Clark measures associated with  $u$  as above except that they need not be singular measures.

**Toeplitz operators.** As mentioned in the introduction, a useful tool in our analysis of the multiplier space  $\mathcal{M}(u, v)$  will be the Toeplitz operators on  $H^2$ . Here for  $\varphi \in L^\infty$ , the essentially bounded Lebesgue measurable functions on  $\mathbb{T}$  with essential supremum norm  $\|\cdot\|_\infty$ , we define the *Toeplitz operator*  $T_\varphi$  with symbol  $\varphi$  by

$$T_\varphi : H^2 \rightarrow H^2, \quad T_\varphi f = P_+(\varphi f),$$

where  $P_+$  is the orthogonal (Riesz) projection from  $L^2$  onto  $H^2$ . Toeplitz operators are well known and well studied [6]. We list a few facts about these operators that will be particularly useful in our analysis of the multipliers  $\mathcal{M}(u, v)$ :

$$(2.24) \quad T_{\bar{z}} = S^*;$$

$$(2.25) \quad \|T_\varphi\| = \|\varphi\|_\infty;$$

$$(2.26) \quad T_\varphi T_\psi = T_{\varphi\psi} \iff \varphi \in \overline{H^\infty} \text{ or } \psi \in H^\infty;$$

$$(2.27) \quad \varphi \in H^\infty \implies \text{Ker } T_{\bar{\varphi}} = \mathcal{K}_{\varphi_i}, \text{ where } \varphi_i \text{ is the inner factor of } \varphi;$$

$$(2.28) \quad \varphi \in H^\infty \implies T_{\bar{\varphi}} \mathcal{K}_u \subseteq \mathcal{K}_u.$$

Observe that (2.28) implies that the model spaces enjoy the so-called *F-property* [49]: If  $w$  is an inner function then

$$(2.29) \quad f \in \mathcal{K}_u \text{ and } \frac{f}{w} \in H^2 \implies \frac{f}{w} \in \mathcal{K}_u.$$

In particular, the outer factor of a function from  $\mathcal{K}_u$  also belongs to  $\mathcal{K}_u$ .

**Hardy space of the upper-half plane.** If  $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$  is the upper-half plane, we set

$$(2.30) \quad \mathcal{H}^2 := \left\{ f \in \mathcal{O}(\mathbb{C}_+) : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx < \infty \right\}$$

to be the Hardy space of the upper-half plane. Via boundary values,  $\mathcal{H}^2$  can be viewed as a (closed) subspace of  $L^2(\mathbb{R})$  and the inner product on  $\mathcal{H}^2$  is the standard  $L^2(\mathbb{R})$  inner product. The corresponding “vanishing Fourier coefficients” characterization of  $\mathcal{H}^2$  via (2.1) is now

$$\mathcal{H}^2 = \{f \in L^2(\mathbb{R}) : \mathcal{F}f|_{(-\infty, 0)} = 0\},$$

where

$$(2.31) \quad (\mathcal{F}f)(\lambda) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \lambda} dx$$

is the Fourier-Plancherel transform. There is an analogous factorization of  $\mathcal{H}^2$  functions into their inner and outer parts. There is also a natural unitary operator  $\mathcal{U} : H^2 \rightarrow \mathcal{H}^2$  given by

$$(2.32) \quad (\mathcal{U}f)(z) := \frac{1}{\sqrt{\pi}(z+i)} f(\omega(z)),$$

where

$$(2.33) \quad \omega(z) := \frac{z-i}{z+i}$$

is the Möbius transform which maps  $\mathbb{C}_+$  onto  $\mathbb{D}$  and  $\mathbb{R} \cup \{-\infty, \infty\}$  onto  $\mathbb{T}$ .

As with  $H^2$ , one can define, for a symbol  $\Psi \in L^\infty(\mathbb{R})$ , the Toeplitz operator

$$T_\Psi : \mathcal{H}^2 \rightarrow \mathcal{H}^2, \quad T_\Psi f = P(\Psi f),$$

where  $P$  is the orthogonal projection of  $L^2(\mathbb{R})$  onto  $\mathcal{H}^2$ .

**Model spaces of the upper-half plane.** For an inner function  $U$  on  $\mathbb{C}_+$  (a bounded analytic function on  $\mathbb{C}_+$  with unimodular boundary values almost everywhere on  $\mathbb{R}$ ), we define the model space

$$(2.34) \quad \mathcal{K}_U := \mathcal{H}^2 \cap (U\mathcal{H}^2)^\perp.$$

The analogue to (2.7) here is the boundary values identity

$$(2.35) \quad \mathcal{K}_\Theta = \mathcal{H}^2 \cap \overline{\Theta \mathcal{H}^2},$$

where  $\overline{\mathcal{H}^2}$  is often regarded as the Hardy space of the lower-half plane. The corresponding reproducing kernel function for  $\mathcal{K}_U$  is

$$K_\lambda^U(z) := \frac{i}{2\pi} \frac{1 - \overline{U(\lambda)}U(z)}{z - \bar{\lambda}}, \quad \lambda, z \in \mathbb{C}_+.$$

As expected, and due to the fact from (2.32) that the operator  $\mathcal{U}$  is unitary, if  $u$  is an inner function on  $\mathbb{D}$  and  $U = u \circ \omega$ , then  $U$  is an inner function on  $\mathbb{C}_+$  (and vice versa). Furthermore, since  $\mathcal{U}$  is unitary we have

$$f \perp_{H^2} uH^2 \iff \mathcal{U}f \perp_{\mathcal{H}^2} U\mathcal{H}^2$$

and thus

$$(2.36) \quad \mathcal{U}\mathcal{K}_u = \mathcal{K}_U.$$

### 3. SOME OBSERVATIONS AND SIMPLIFICATIONS

In this section we recall some (essentially) known results for which we provide short proofs that are new variations to known ones. We feel this will help clarify certain parts of our upcoming discussions.

**The multipliers of  $H^2$ .** To give us a reference for our results on model spaces, let us begin with the following well-known characterization of  $\mathcal{M}(H^2, H^2)$ . We include a proof in order to point out the difficulties when trying to characterize the multipliers between two model spaces.

**Proposition 3.1.**  $\mathcal{M}(H^2, H^2) = H^\infty$ .

*Proof.* Using the integral means characterization of  $H^2$  from (2.2), we obtain  $H^\infty \subseteq \mathcal{M}(H^2, H^2)$ . For the reverse inclusion, note that for  $\varphi \in \mathcal{M}(H^2, H^2)$  an application of the closed graph theorem shows that the multiplication (Laurent) operator

$$M_\varphi : H^2 \rightarrow H^2, \quad M_\varphi f = \varphi f$$

is bounded. The reproducing property of the Cauchy kernel from (2.5) yields

$$(3.2) \quad (M_\varphi^* k_\lambda)(z) = \langle M_\varphi^* k_\lambda, k_z \rangle = \langle k_\lambda, \varphi k_z \rangle = \overline{(\varphi k_z)(\lambda)} = \overline{\varphi(\lambda)} k_\lambda(z),$$

which implies that the range of  $\varphi$  is contained in the bounded set  $\overline{\sigma(M_\varphi^*)}$ , where  $\sigma(M_\varphi^*)$  is the spectrum of  $M_\varphi^*$  and the “bar” denotes complex conjugation.  $\square$

**Remark 3.3.** For most “reasonable” reproducing kernel Hilbert spaces  $\mathcal{H}$  of analytic functions on  $\mathbb{D}$ , the eigenvalue identity in (3.2) will show that  $\mathcal{M}(\mathcal{H}, \mathcal{H}) \subseteq H^\infty$  [26, Corollary 9.7]. This process breaks down when examining the multipliers  $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$  for two *different* Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$ . For example, the argument used in (3.2) will show that when  $\varphi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ , the adjoint of the multiplication operator  $M_\varphi : \mathcal{H}_1 \rightarrow \mathcal{H}_2, M_\varphi f = \varphi f$ , satisfies

$$(3.4) \quad M_\varphi^* k_\lambda^{\mathcal{H}_2} = \overline{\varphi(\lambda)} k_\lambda^{\mathcal{H}_1}, \quad \lambda \in \mathbb{D},$$

where  $k_\lambda^{\mathcal{H}_j}, j = 1, 2$ , are the corresponding reproducing kernels. However, since the spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are different, as are their reproducing kernels,  $\overline{\varphi(\lambda)}$  will not be an eigenvalue of  $M_\varphi^*$ . Despite this difficulty, one can use (3.4) to show that

$$(3.5) \quad |\varphi(\lambda)| \leq \|M_\varphi^*\|_{\mathcal{H}_2 \rightarrow \mathcal{H}_1} \frac{\|k_\lambda^{\mathcal{H}_2}\|_{\mathcal{H}_2}}{\|k_\lambda^{\mathcal{H}_1}\|_{\mathcal{H}_1}}, \quad \lambda \in \mathbb{D}.$$

Unfortunately, the norms of the two kernels appearing in (3.5) may behave quite differently as  $|\lambda| \rightarrow 1$  and thus this argument cannot be used to show that  $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \subseteq H^\infty$ . There is a very good reason why we run into trouble. Indeed, as we will see in Section 8 below, the multiplier space  $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$  may contain unbounded functions. Although, when  $\|k_\lambda^{\mathcal{H}_2}\|_{\mathcal{H}_2} \lesssim \|k_\lambda^{\mathcal{H}_1}\|_{\mathcal{H}_1}, \lambda \in \mathbb{D}$ , we do indeed obtain  $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \subseteq H^\infty$ .

Further complicating matters is the fact that the containment  $H^\infty \subseteq \mathcal{M}(H^2, H^2)$  came from the growth condition (2.2) which defined  $H^2$ . For other spaces, such as the classical Dirichlet space  $\mathcal{D}$ , where membership is determined by a different growth condition (derivative and integral), not every bounded function is a multiplier [23, 50].

When examining the multipliers  $\mathcal{M}(u, v)$  between the model spaces  $\mathcal{K}_u$  and  $\mathcal{K}_v$ , we encounter *both* of these difficulties. Indeed, since  $\mathcal{K}_u$  is described by the orthogonality condition  $\mathcal{K}_u = (uH^2)^\perp$ , and not a growth condition as with  $H^2$ , we can’t automatically conclude that  $H^\infty \subseteq \mathcal{M}(u, v)$ . Furthermore, as we will see in Section 8, we don’t automatically get the other inclusion  $\mathcal{M}(u, v) \subseteq H^\infty$ .

**Multipliers between model spaces.** Let us now formally define our notation for multipliers between model spaces. For two inner functions  $u$  and  $v$ , let

$$\mathcal{M}(u, v) := \{\varphi \in \mathcal{O}(\mathbb{D}) : \varphi \mathcal{K}_u \subseteq \mathcal{K}_v\}$$

denote the *multipliers* from  $\mathcal{K}_u$  into  $\mathcal{K}_v$  and

$$\mathcal{M}_b(u, v) := \mathcal{M}(u, v) \cap H^\infty$$

denote the bounded multipliers from  $\mathcal{K}_u$  into  $\mathcal{K}_v$ . Though these two multiplier spaces will be the two main objects of our study, we will also have a few thoughts (see Theorem 4.11 below) on the  $\varphi \in \mathcal{M}(u, v)$  for which

$$(3.6) \quad \|\varphi f\| = \|f\|, \quad f \in \mathcal{K}_u,$$

in other words, the *isometric multipliers* from  $\mathcal{K}_u$  into  $\mathcal{K}_v$ , denoted by  $\mathcal{M}_i(u, v)$ .

We begin with some simple but very useful observations. First notice that

$$(3.7) \quad \mathcal{M}(u, v) \subseteq H^2.$$

Indeed, as discussed earlier,  $k_0^u = 1 - \overline{u(0)}u \in \mathcal{K}_u$  and is an invertible element of  $H^\infty$ . Thus if  $\varphi \in \mathcal{M}(u, v)$  then  $\varphi k_0^u \in \mathcal{K}_v \subseteq H^2$  from which (3.7) follows.

By a similar computation as in (3.4) we have  $M_\varphi^* : \mathcal{K}_v \rightarrow \mathcal{K}_u$  and

$$(3.8) \quad M_\varphi^* k_\lambda^v = \overline{\varphi(\lambda)} k_\lambda^u, \quad \lambda \in \mathbb{D}.$$

Since

$$\|k_\lambda^u\|^2 = \langle k_\lambda^u, k_\lambda^u \rangle = k_\lambda^u(\lambda) = \frac{1 - |u(\lambda)|^2}{1 - |\lambda|^2},$$

we see from (3.5) that

$$(3.9) \quad |\varphi(\lambda)|^2 (1 - |u(\lambda)|^2) \lesssim (1 - |v(\lambda)|^2), \quad \lambda \in \mathbb{D}.$$

Unfortunately this inequality does not prove that  $\varphi$  is bounded. However, it does yield some useful control on the pointwise growth of a multiplier as  $|\lambda| \rightarrow 1$ . This estimate will be exploited in Example 4.7 and again in the proof of Corollary 5.2.

The following result from Crofoot [14, Proposition 12] says that the multipliers from a model space to *itself* are not a very interesting class of functions.

**Proposition 3.10.**  $\mathcal{M}(u, u) = \mathbb{C}$ .

*Proof.* For completeness we provide a variation of the proof found in [27] (and slightly different in flavor from Crofoot's proof).

The inclusion  $\mathbb{C} \subseteq \mathcal{M}(u, u)$  is automatic. To prove the converse, we will use the fact that  $S^*u \in \mathcal{K}_u$  (from (2.13)) and show that  $\varphi S^*u \notin \mathcal{K}_u$  for

every  $\varphi \in H^\infty \setminus \mathbb{C}$  (Note that it follows from (3.9) that if  $\varphi \in \mathcal{M}(u, u)$  then  $\varphi \in H^\infty$ ).

Suppose towards a contradiction that  $\varphi S^*u \in \mathcal{K}_u$  and  $\varphi \in H^\infty \setminus \mathbb{C}$ . Then

$$\begin{aligned} 0 &= T_{\bar{u}}(\varphi S^*u) \quad (\text{by (2.27)}) \\ &= P_+ \left( \bar{u} \varphi \frac{u - u(0)}{z} \right) \\ &= P_+(\bar{z} \varphi (1 - u(0)\bar{u})) \\ &= T_{1-u(0)\bar{u}} T_{\bar{z}} \varphi \quad (\text{by (2.26)}) \\ &= T_{1-u(0)\bar{u}}(S^* \varphi) \quad (\text{by (2.24)}), \end{aligned}$$

which we rewrite as

$$S^* \varphi = u(0) T_{\bar{u}} S^* \varphi.$$

Using the identity  $\|T_{\bar{u}}\| = \|T_u\| = \|u\|_\infty = 1$  from (2.25), we deduce the inequality

$$\|S^* \varphi\| \leq |u(0)| \|S^* \varphi\|.$$

Since  $\varphi \in H^\infty \setminus \mathbb{C}$ , we see that  $\|S^* \varphi\| \neq 0$ . Therefore,  $|u(0)| \geq 1$ , which, by the Maximum Modulus Theorem, forces  $u$  to be a unimodular constant (which it is not). This produces the desired contradiction.  $\square$

**Proposition 3.11.**  $\mathbb{C} \subseteq \mathcal{M}(u, v)$  if and only if  $u$  divides  $v$ .

*Proof.* It is easy to see that  $\mathbb{C} \subseteq \mathcal{M}(u, v)$  if and only if  $\mathcal{K}_u \subseteq \mathcal{K}_v$ . By (2.10) this is equivalent to the fact that  $u$  divides  $v$ .  $\square$

**Proposition 3.12.** Suppose  $u$  divides  $v$  and  $u$  is not a constant multiple of  $v$ . Then  $\mathcal{M}(v, u) = \{0\}$ .

*Proof.* If  $\varphi \in \mathcal{M}(v, u)$  then, since  $\mathcal{K}_u \subseteq \mathcal{K}_v$  (2.10), we get

$$\varphi \mathcal{K}_u \subseteq \varphi \mathcal{K}_v \subseteq \mathcal{K}_u.$$

This means that  $\varphi \equiv c$  (Proposition 3.10) and so  $c \mathcal{K}_v \subseteq \mathcal{K}_u$ . Hence, if  $c \neq 0$  we can use (2.10) again to see that  $v$  divides  $u$ , i.e.,  $u$  is a constant multiple of  $v$  which we excluded. Thus  $c = 0$ .  $\square$

**Remark 3.13.** In our analysis below, we will make use of the following simple observation (also found in Crofoot's paper [14, Cor. 4]): If  $\varphi \mathcal{K}_u = \mathcal{K}_v$ , then the map  $f \mapsto \varphi f$  is an isomorphism whose inverse is  $g \mapsto \frac{1}{\varphi} g$ . Moreover, the analysis used to prove (3.7) shows that  $\frac{1}{\varphi} \in H^2$ . This also shows that  $\varphi$  must be outer [29, p. 65] (see also Remark 3.24 below).

**The Crofoot transform.** This next result of Crofoot [14] has proven to be a useful tool in studying model spaces (see [27, Ch. 13]). For the sake of completeness, we include its short proof.

**Theorem 3.14** (Crofoot). *If  $u$  is inner,  $a \in \mathbb{D}$ , and*

$$(3.15) \quad u_a := \frac{u - a}{1 - \bar{a}u}$$

*is a Frostman shift for  $u$ , then*

$$\frac{1}{1 - \bar{a}u} \mathcal{K}_u = \mathcal{K}_{u_a}.$$

*Proof.* By evaluating on  $\mathbb{T}$ , one can see that  $u_a$  is indeed an inner function. For  $f \in \mathcal{K}_u$  we see from (2.7) that  $f = u\bar{z}g$  almost everywhere on  $\mathbb{T}$  for some  $g \in H^2$ . Thus, almost everywhere on  $\mathbb{T}$ , we have

$$\frac{1}{1 - \bar{a}u} f = \frac{u - a}{1 - \bar{a}u} \overline{\left( z \frac{g}{1 - \bar{a}u} \right)} \in u_a \overline{zH^2}.$$

This yields

$$(3.16) \quad \frac{1}{1 - \bar{a}u} \mathcal{K}_u \subseteq \mathcal{K}_{u_a}.$$

For the reverse inclusion, observe that since

$$u = \frac{u_a + a}{1 + \bar{a}u_a},$$

an analogous argument to the one above will yield

$$\frac{1}{1 + \bar{a}u_a} \mathcal{K}_{u_a} \subseteq \mathcal{K}_u.$$

Now multiply both sides of the previous inclusion by  $(1 - \bar{a}u)^{-1}$  and use the identity

$$(1 - \bar{a}u)(1 + \bar{a}u_a) = 1 - |a|^2$$

along with (3.16) to obtain the result.  $\square$

**Remark 3.17.** (1) With a little more work, one can show that the map

$$f \mapsto \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}u} f,$$

sometimes called the “Crofoot transform” is *isometric* from  $\mathcal{K}_u$  onto  $\mathcal{K}_{u_a}$ .



- (2) Crofoot's theorem says that when examining  $\mathcal{M}(u, v)$ , we can, if needed, assume that one (or both) of  $u(0)$  or  $v(0)$  vanish. Putting this in more precise notation:

$$(3.18) \quad \frac{1 - \bar{\lambda}u}{1 - \bar{\eta}v} \mathcal{M}(u, v) = \mathcal{M}(u_\lambda, v_\eta).$$

In fact we also have

$$(3.19) \quad \frac{1 - \bar{\lambda}u}{1 - \bar{\eta}v} \mathcal{M}_b(u, v) = \mathcal{M}_b(u_\lambda, v_\eta).$$

**Outer multipliers.** The set of multipliers is closed under removing inner factors.

**Proposition 3.20.** *If  $\varphi \in \mathcal{M}(u, v)$  and  $F$  is the outer factor of  $\varphi$ , then  $F \in \mathcal{M}(u, v)$ .*

*Proof.* Using the  $F$ -property for model spaces (2.29) we see that if  $w$  is the inner factor of  $\varphi$  then

$$\varphi f \in \mathcal{K}_v \quad \forall f \in \mathcal{K}_u \implies \frac{\varphi}{w} f \in \mathcal{K}_v \quad \forall f \in \mathcal{K}_u,$$

which implies that the outer function  $F = \varphi/w$  belongs to  $\mathcal{M}(u, v)$ .  $\square$

Thus  $\mathcal{M}(u, v)$  contains an outer function whenever  $\mathcal{M}(u, v) \neq \{0\}$ .

The next basic fact says that if there is an onto multiplier, there is essentially only one multiplier [14, Corollary 13] and it must be outer [14, Corollary 4].

**Proposition 3.21** (Crofoot). *If  $\varphi \mathcal{K}_u = \mathcal{K}_v$ , then  $\mathcal{M}(u, v) = \mathbb{C}\varphi$ . Furthermore,  $\varphi$  is an outer function.*

*Proof.* The proof of this result can be found in Crofoot's paper. However it follows from our discussion above. To see that  $\varphi$  is outer note that  $k_0^v = 1 - \overline{v(0)}v$  is outer (any function of the form  $1 + q$ , where  $q(\mathbb{D}) \subseteq \mathbb{D}$ , is outer [29, p. 65]). Thus since  $\varphi \mathcal{K}_u = \mathcal{K}_v$  we see that  $k_0^v = \varphi h$  for some  $h \in \mathcal{K}_u$ . It follows from factorization in  $H^2$  (see (2.3)) that  $\varphi$  is outer.

To see that  $\mathcal{M}(u, v) = \mathbb{C}\varphi$  we can use Proposition 3.10. Indeed, if  $\psi \in \mathcal{M}(u, v)$  then

$$\frac{\psi}{\varphi} \mathcal{K}_u \subseteq \frac{1}{\varphi} \mathcal{K}_v = \mathcal{K}_u,$$

whence  $\psi/\varphi$  must be a constant function. Thus  $\mathcal{M}(u, v) = \mathbb{C}\varphi$ .  $\square$

**Corollary 3.22.** *If  $\mathcal{M}(u, v) \neq \{0\}$  and  $\mathcal{M}(v, u) \neq \{0\}$ , then there exists a  $\varphi \in H^2$  with  $\varphi\mathcal{K}_u = \mathcal{K}_v$  and  $\mathcal{M}(u, v) = \mathbb{C}\varphi$ .*

*Proof.* Let  $\varphi \in \mathcal{M}(u, v) \setminus \{0\}$  and  $\psi \in \mathcal{M}(v, u) \setminus \{0\}$ . Then

$$\psi\varphi\mathcal{K}_u \subseteq \psi\mathcal{K}_v \subseteq \mathcal{K}_u,$$

and Proposition 3.10 implies that  $\psi\varphi$  is a non-zero constant. It now follows that  $\varphi\mathcal{K}_u = \mathcal{K}_v$ . To finish, apply Proposition 3.21.  $\square$

The last corollary immediately yields the following.

**Corollary 3.23.** *If  $\dim \mathcal{M}(u, v) \geq 2$  then  $\mathcal{M}(v, u) = \{0\}$ .*

**Remark 3.24.** We pause here for another observation. By Remark 3.13, if the multiplier is onto, then  $\varphi$  and  $1/\varphi$  belong to  $H^2$ . Though we will not develop this much further here, this condition implies that (but is not equivalent to)  $\varphi^2$  is a “rigid function” in  $H^1$  meaning that  $\varphi^2$  is uniquely determined by its argument. See [46] for more on this. Let us state this observation as a separate result.

**Corollary 3.25.** *If  $\varphi$  is an onto multiplier between two model spaces, then  $\varphi^2$  is rigid in  $H^1$ .*

**Finite Blaschke products.** We now use another idea found in Crofoot’s paper [14] which we will develop further in Theorem 4.10.

**Definition 3.26.** For an inner function  $u$  define the *degree* of  $u$  to be  $n$  if  $u$  is a finite Blaschke product with  $n$  zeros (repeated according to their multiplicity) and equal to  $\infty$  otherwise.

So, for example, any Blaschke product with infinitely many zeros is of infinite degree as is any singular inner function.

**Proposition 3.27.** *Suppose that  $u$  and  $v$  are two finite Blaschke products with  $\deg(u) \leq \deg(v)$ . Then  $\mathcal{M}(u, v) = \mathcal{M}_b(u, v) \neq \{0\}$ .*

*Proof.* If  $m \leq n$  and  $\{a_1, \dots, a_m\}$ ,  $\{b_1, \dots, b_n\}$  are the respective zeros of  $u$  and  $v$  (repeated according to their multiplicity), then by (2.19) we have

$$\begin{aligned} \mathcal{K}_u &= \left\{ \frac{p(z)}{\prod_{j=1}^m (1 - \overline{a_j}z)} : p \in \mathcal{P}_{m-1} \right\}, \\ \mathcal{K}_v &= \left\{ \frac{p(z)}{\prod_{j=1}^n (1 - \overline{b_j}z)} : p \in \mathcal{P}_{n-1} \right\}. \end{aligned}$$

If

$$\varphi(z) := \frac{\prod_{j=1}^m (1 - \overline{a_j} z)}{\prod_{j=1}^n (1 - \overline{b_j} z)},$$

then  $\varphi\mathcal{K}_u \subseteq \mathcal{K}_v$  and so  $\mathcal{M}_b(u, v) \neq \{0\}$ .

To see that  $\mathcal{M}(u, v) = \mathcal{M}_b(u, v)$  notice that if  $\varphi \in \mathcal{M}(u, v)$  then  $\varphi k_0^u \in \mathcal{K}_v$ . But since  $\mathcal{K}_v \subseteq H^\infty$  and  $k_0^u$  is invertible in  $H^\infty$  then  $\varphi \in H^\infty$ .  $\square$

There is a version of this result for certain types of infinite Blaschke products in [14, Theorem 27] where the focus is the onto multipliers.

By just considering the dimensions of the spaces  $\mathcal{K}_u$  and  $\mathcal{K}_v$ , one concludes that  $\mathcal{M}(u, v) = \{0\}$  when  $u$  and  $v$  are finite Blaschke products with  $\deg(u) > \deg(v)$ . Once we have developed a few more technical tools, we will describe  $\mathcal{M}(u, v)$  when  $u$  and  $v$  are finite Blaschke products (see Theorem 4.10 below).

**Corollary 3.28.** *If  $u$  is a finite Blaschke product,  $v$  is a finite Blaschke product with  $\deg(u) \leq \deg(v)$ , and  $w$  is any other inner function. Then  $\mathcal{M}(u, vw) \neq \{0\}$ .*

An extension of this is the following.

**Theorem 3.29.** *If  $u$  is a finite Blaschke product and  $v$  is any inner function with infinite degree, then  $\mathcal{M}_b(u, v) \neq \{0\}$ .*

*Proof.* By a classical theorem of Frostman [29, p. 75] there is an  $a \in \mathbb{D}$  (in fact many such points  $a$ ) such that the Frostman shift  $v_a$  of  $v$  from (1.3) is a Blaschke product of infinite degree. Factor  $v_a = IJ$ , where  $I$  and  $J$  are Blaschke products with the degree of  $I$  equal to the degree of  $u$ , and observe from (2.10) that  $\mathcal{K}_I \subseteq \mathcal{K}_{v_a}$ .

From Proposition 3.27 there is a  $\varphi \in H^\infty$  such that

$$\varphi\mathcal{K}_u \subseteq \mathcal{K}_I \subseteq \mathcal{K}_{v_a}.$$

Now use Crofoot's theorem (Theorem 3.14) to get

$$(1 - \overline{a}v)\varphi\mathcal{K}_u \subseteq \mathcal{K}_v. \quad \square$$

**Multipliers and boundary spectra.** The next result concerning the boundary spectra is the “into-multiplier” version of [14, Theorem 14]. Crofoot shows equality of spectra in the situation of “onto-multipliers” using operator theory techniques. Our proof is based on function theory. Recall the boundary spectrum  $\sigma(u)$  from (2.14).

**Proposition 3.30.** *If  $\mathcal{M}(u, v) \neq \{0\}$  then  $\sigma(u) \subseteq \sigma(v)$ .*

*Proof.* Without loss of generality, we can use Crofoot's theorem (Theorem 3.14) and assume that  $u(0) = 0$  (the Crofoot transform preserves the regular points in  $\mathbb{T}$ ). By (2.6), the constant function 1 belongs to  $\mathcal{K}_u$  and so

$$\varphi \mathcal{K}_u \subseteq \mathcal{K}_v \implies \varphi \in \mathcal{K}_v.$$

Pick  $\zeta \in \mathbb{T} \setminus \sigma(v)$  (a regular point for  $v$ ). By (2.15), every function in  $\mathcal{K}_v$  has an analytic continuation to a two-dimensional open neighborhood  $\Omega$  of  $\zeta$ . In particular,  $\varphi \in \mathcal{K}_v$  enjoys this property. For every  $f \in \mathcal{K}_u$ ,  $g := \varphi f \in \mathcal{K}_v$  has an analytic continuation to  $\Omega$  and so  $f = g/\varphi$  is either analytic on  $\Omega$  or has a pole of order at least 1 at  $\zeta$ . But this second case is not possible since  $f \in H^2$  must be square integrable on  $\mathbb{T}$ . Hence  $f$  extends analytically to  $\Omega$  and thus  $\zeta \in \mathbb{T} \setminus \sigma(u)$ .  $\square$

The above result shows, for example, that when  $u$  and  $v$  are the atomic inner functions

$$u(z) = \exp\left(-\frac{\xi_1 + z}{\xi_1 - z}\right), \quad v(z) = \exp\left(-\frac{\xi_2 + z}{\xi_2 - z}\right), \quad \xi_1 \neq \xi_2,$$

(note that  $\sigma(u) = \{\xi_1\}$ ,  $\sigma(v) = \{\xi_2\}$  from (2.16)) we have  $\mathcal{M}(u, v) = \{0\}$ .

**Remark 3.31.** (1) Note that the converse of Proposition 3.30 is not true. Indeed, if  $u = vI$ , where  $I$  is a non-constant finite Blaschke product, then  $\sigma(u) = \sigma(v)$  (finite Blaschke products have analytic continuations across all of  $\mathbb{T}$ ). However, by Proposition 3.12 we know that  $\mathcal{M}(u, v) = \{0\}$ .

(2) By (3.9) and (2.17) it seems natural to conjecture that there is an analogue to Proposition 3.30 for the Ahern-Clark points of  $u$  and  $v$  (see (2.17)). However, we will see in Proposition 7.16 that it is possible for  $\mathcal{M}(u, v)$  to be non-zero,  $\sigma(u) = \sigma(v)$ , but the Ahern-Clark points for  $u$  to not be contained in the Ahern-Clark points for  $v$ .

#### 4. THE MAIN RESULT

We will now state and prove the main result of this paper which characterizes multipliers in terms of kernels of Toeplitz operators and Carleson measures. Recall the definition of a Carleson measure for  $\mathcal{K}_u$  from (1.4) and also recall that if  $\varphi \in H^2$  then  $|\varphi|^2 dm$  is a Carleson measure for

$\mathcal{K}_u$  if and only if  $\varphi \in \mathcal{M}(\mathcal{K}_u, H^2)$ . We now focus on the extra condition needed to ensure that  $\varphi \in \mathcal{M}(u, v)$ .

**Theorem 4.1.** *For inner functions  $u$  and  $v$  and  $\varphi \in H^2$ , the following are equivalent:*

- (i)  $\varphi \in \mathcal{M}(u, v)$ ;
- (ii)  $\varphi S^*u \in \mathcal{K}_v$  and  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ .
- (iii)  $\varphi \in \text{Ker } T_{\overline{v}u}$  and  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ .

Furthermore, the following are equivalent:

- (iv)  $\varphi \in \mathcal{M}_b(u, v)$ ;
- (v)  $\varphi S^*u \in \mathcal{K}_v \cap H^\infty$ .
- (vi)  $\varphi \in \text{Ker } T_{\overline{v}u} \cap H^\infty$ .

Before proving Theorem 4.1, would like to emphasize the following immediate consequence:

**Corollary 4.2.**  $\text{Ker } T_{\overline{v}u} \cap H^\infty = \mathcal{M}_b(u, v) \subseteq \mathcal{M}(u, v) \subseteq \text{Ker } T_{\overline{v}u}$ .

As an easy example of this theorem, let  $u$  be an inner function and  $v = z^N u$ . Then

$$\text{Ker } T_{\overline{v}u} = \text{Ker } T_{\overline{z}^{N+1}} = \mathcal{P}_N.$$

So in this case we have

$$\mathcal{M}_b(u, v) = \mathcal{M}(u, v) = \mathcal{P}_N.$$

We will see in Example 4.7 below that, in general,  $\mathcal{M}(u, v) \subsetneq \text{Ker } T_{\overline{v}u}$ .

*Proof of Theorem 4.1.* First observe the equivalences

$$\begin{aligned} \varphi S^*u \in \mathcal{K}_v &\iff T_{\overline{v}}(\varphi S^*u) = 0 \quad (\text{by (2.27)}) \\ &\iff T_{\overline{v}S^*u}\varphi = 0 \quad (\text{by (2.26)}) \\ &\iff \varphi \in \text{Ker } T_{\overline{v}S^*u}. \end{aligned}$$

Moreover,

$$T_{\overline{v}S^*u} = T_{\overline{v}(u-u(0))} = T_{\overline{v}u(1-u(0))\overline{u}} = T_{\overline{1-u(0)u}}T_{\overline{v}u}.$$

Observe that  $T_{\overline{1-u(0)u}}$  is an invertible Toeplitz operator and thus

$$\varphi S^*u \in \mathcal{K}_v \iff \varphi \in \text{Ker } T_{\overline{v}u}.$$

This yields (ii)  $\iff$  (iii) and (vi)  $\implies$  (v). The reverse implication (v)  $\implies$  (vi) needs an additional argument. Indeed, suppose that

$\varphi S^*u \in \mathcal{K}_v \cap H^\infty$ . Then the above equivalences yield  $\varphi \in \text{Ker } T_{\overline{v}u}$ , and we just have to check that  $\varphi$  is bounded. We already know that  $\varphi \in H^2$ . Thus in order to verify  $\varphi \in H^\infty$ , it suffices to prove that  $\varphi|_{\mathbb{T}} \in L^\infty$  (Smirnov's theorem [17, p. 28]). By assumption,  $\varphi S^*u = g \in H^\infty$  and thus

$$g = \varphi S^*u = \varphi \overline{z}(u - u(0)) = \varphi u \frac{1 - u(0)\overline{u}}{z}.$$

On  $\mathbb{T}$  we have

$$|\varphi| = \left| \frac{g}{1 - \overline{u(0)}u} \right|$$

which is uniformly bounded since  $|u(0)| < 1$ . Hence  $\varphi \in H^\infty$  which proves  $(v) \implies (vi)$ .

We now prove  $(i) \implies (ii)$ . As mentioned already, when  $\varphi \in \mathcal{M}(u, v)$  then  $|\varphi|^2 dm$  is necessarily a Carleson measure for  $\mathcal{K}_v$ . Furthermore, since  $S^*u \in \mathcal{K}_u$ , we have  $\varphi S^*u \in \mathcal{K}_v$ .

The implication  $(iv) \implies (v)$  is now automatic since  $\varphi \in \mathcal{M}_b(u, v)$ , by definition, is bounded.

For the implication  $(v) \implies (iv)$ , it remains to observe that if  $\varphi \in H^\infty$  then  $|\varphi|^2 dm$  is automatically a Carleson measure, and the result follows once we have shown  $(ii) \implies (i)$ .

So it remains to prove  $(ii) \implies (i)$ . We will use (2.13) which asserts that the linear span of  $\{S^{*n}u : n \geq 1\}$  forms a dense set in  $\mathcal{K}_u$ . Our first step will be to show that  $\varphi S^{*n}u \in \mathcal{K}_v$ ,  $n \geq 2$ , or equivalently, via (2.26),  $T_{\overline{v}}(\varphi S^{*n}u) = 0$ , when  $\varphi S^*u \in \mathcal{K}_v$ . By using (2.7) we observe the following equivalences:

$$\begin{aligned} \varphi S^*u \in \mathcal{K}_v &\iff \varphi \overline{z}(u - u(0)) = v\overline{\psi}, \quad \psi \in zH^2 \\ &\iff \varphi u(1 - u(0)\overline{u}) = zv\overline{\psi}, \quad \psi \in zH^2 \\ &\iff \varphi = zv \overline{\left( \frac{u\psi}{1 - \overline{u(0)}u} \right)}, \quad \psi \in zH^2 \end{aligned}$$

In the above recall that  $1 - \overline{u(0)}u$  is an outer function [29, p. 65] and invertible in  $H^\infty$ . By the formula

$$S^{*n}u = \frac{1}{z^n} \left( u - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} z^k \right),$$

we have  $S^{*n}u = \overline{z^n}(u - p)$  on  $\mathbb{T}$ , where  $p$  is an analytic polynomial of degree at most  $n - 1$ . Hence when  $n \geq 2$ ,

$$\begin{aligned} T_{\overline{v}}(\varphi S^{*n}u) &= P_+ \left[ \overline{vzv \left( \frac{u\psi}{1 - \overline{u(0)u}} \right)} \left( \overline{z^n}(u - p) \right) \right] \\ &= P_+ \left[ \left( \overline{z^{n-1}}(u - p) \right) \overline{\left( \frac{u\psi}{1 - \overline{u(0)u}} \right)} \right] \\ &= P_+ \left[ \overline{z^{n-1} \left( \frac{\psi}{1 - \overline{u(0)u}} \right)} \right] - P_+ \left[ \overline{z^{n-1}p \left( \frac{u\psi}{1 - \overline{u(0)u}} \right)} \right]. \end{aligned}$$

The first term in the above summand vanishes since

$$z^{n-1} \cdot \frac{\psi}{1 - \overline{u(0)u}} \in zH^2.$$

The second term also vanishes for a similar reason in that the degree of  $p$  is at most  $n - 1$  and  $\psi(0) = 0$ . As a result,

$$(4.3) \quad \varphi S^{*n}u \in \mathcal{K}_v \quad \forall n \geq 1.$$

It is here that we use the fact that  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$  in that  $f \mapsto \varphi f$  defines a bounded operator from  $\mathcal{K}_u$  to  $H^2$  and thus the inclusion

$$\varphi \cdot \text{span}\{S^{*n}u : n \geq 1\} \subseteq \mathcal{K}_v$$

from (4.3) extends to the inclusion  $\varphi\mathcal{K}_u \subseteq \mathcal{K}_v$ .  $\square$

Recall from Proposition 3.11 that if  $u$  divides  $v$  then  $\mathbb{C} \subseteq \mathcal{M}(u, v)$ . The above result allows us to say more.

**Corollary 4.4.** *Suppose  $u$  is inner and  $v = uI$  for some inner  $I$ . Then*

$$\mathcal{M}_b(u, v) = \mathcal{K}_{zI} \cap H^\infty \subseteq \mathcal{M}(u, v) \subseteq \mathcal{K}_{zI}.$$

*Proof.* Clearly  $\overline{z}v u = \overline{z}I$  on  $\mathbb{T}$ .  $\square$

We can actually go one step further and characterize  $\mathcal{M}(u, v)$  when  $v = uI$ .

**Corollary 4.5.** *Suppose  $u$  and  $v$  are inner functions and  $v = uI$ . Then the following are equivalent:*

- (i)  $\varphi \in \mathcal{M}(u, v)$ ;
- (ii)  $\varphi \in \mathcal{K}_{zI}$  and  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ .

*Furthermore, the following are equivalent:*

(iii)  $\varphi \in \mathcal{M}_b(u, v)$ ;

(iv)  $\varphi \in \mathcal{K}_{zI} \cap H^\infty$ .

Finally, if  $I$  is a finite Blaschke product then

$$\mathcal{M}_b(u, v) = \mathcal{M}(u, v) = \mathcal{K}_{zI}.$$

**Remark 4.6.** A little thought using the definition of the model spaces will show that

$$(\mathcal{K}_I \cap H^\infty) \cdot \mathcal{K}_u \subseteq \mathcal{K}_{uI}.$$

It follows from this observation and the previous discussion that every  $\varphi \in \mathcal{K}_I$  for which  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$  is a multiplier from  $\mathcal{K}_u$  to  $\mathcal{K}_v$ . However, our analysis shows that this does not exhaust the whole multiplier space (which has one more dimension).

**Multipliers and kernels.** We now construct an example of when  $\text{Ker } T_{zI} = \text{Ker } T_{\overline{z}vu}$  contains functions which do not define Carleson measures for  $\mathcal{K}_u$  and thus

$$\mathcal{M}(u, v) \subsetneq \text{Ker } T_{\overline{z}vu}.$$

Hence the Carleson condition is important in Theorem 4.1.

**Example 4.7.** Set  $\lambda_n = 1 - 2^{-n}$ ,  $n \geq 1$ , and note that  $\{\lambda_n\}_{n \geq 1}$  is the zero sequence of an interpolating Blaschke product  $I$ . With  $w_n = n^{-1}$ , notice that  $\sum_{n \geq 1} w_n^2 < \infty$ . By an interpolation theorem from [40, p. 135], there is a  $\varphi \in \mathcal{K}_I \subseteq \mathcal{K}_{zI}$  such that

$$(4.8) \quad \varphi(\lambda_n) = \frac{w_n}{(1 - |\lambda_n|^2)^{1/2}} \asymp \frac{2^{n/2}}{n} \rightarrow \infty.$$

Now take  $u$  to be the inner function

$$u(z) = \exp\left(\frac{z+1}{z-1}\right)$$

and observe that since  $\lambda_n \rightarrow 1$  on  $(0, 1)$  we have

$$(4.9) \quad u(\lambda_n) \rightarrow 0.$$

If  $v = uI$  then

$$\varphi \in \mathcal{K}_I \subseteq \mathcal{K}_{zI} = \text{Ker } T_{\overline{z}vu}.$$

However,  $\varphi \notin \mathcal{M}(u, v)$  since, if it were, (3.9) would imply that

$$|\varphi(\lambda_n)|^2 (1 - |u(\lambda_n)|^2) \lesssim 1 - |v(\lambda_n)|^2 \lesssim 1.$$

By (4.8) and (4.9), the left-hand side of the above string of inequalities approaches  $+\infty$  as  $n \rightarrow \infty$  which yields a contradiction. So in this case we have  $\mathcal{M}(u, v) \subsetneq \text{Ker } T_{\overline{z}vu} = \mathcal{K}_{zI}$ .



**Finite Blaschke products again.** We now take another look at the finite dimensional case treated in Proposition 3.27 and, in light of Theorem 4.1, give a very tangible description of the multipliers.

**Theorem 4.10.** *If  $u$  is a finite Blaschke product with zeros  $\{a_1, \dots, a_m\}$  and  $v$  is a finite Blaschke product with zeros  $\{b_1, \dots, b_n\}$  where  $m \leq n$ , and the zeros are repeated according to their multiplicity, then*

$$\mathcal{M}(u, v) = \mathcal{M}_b(u, v) = \left\{ q(z) \frac{\prod_{i=1}^m (1 - \overline{a_i} z)}{\prod_{j=1}^n (1 - \overline{b_j} z)} : q \in \mathcal{P}_{n-m} \right\}.$$

*Proof.* The  $\supseteq$  containment follows from (2.19) and the proof of Proposition 3.27. For the  $\subseteq$  containment, notice from Theorem 4.1 that

$$\varphi \in \mathcal{M}(u, v) \implies \varphi \in \text{Ker } T_{\overline{v}u}$$

which is equivalent to

$$u\varphi \in \text{Ker } T_{\overline{v}} = \mathcal{K}_{zv} = \left\{ \frac{p(z)}{\prod_{j=1}^n (1 - \overline{b_j} z)} : p \in \mathcal{P}_n \right\} \subseteq H^\infty.$$

Hence

$$\varphi = \frac{p}{\prod_{j=1}^n (1 - \overline{b_j} z)} \frac{1}{u} = \frac{\prod_{j=1}^m (1 - \overline{a_j} z)}{\prod_{j=1}^n (1 - \overline{b_j} z)} \cdot \frac{p}{\prod_{j=1}^m (z - a_j)},$$

for some polynomial  $p$  of degree at most  $n$ . But since  $\varphi$  must be analytic on  $\mathbb{D}$  we must have  $p(a_j) = 0$  for all  $1 \leq j \leq m$  and thus

$$\varphi = \frac{\prod_{j=1}^m (1 - \overline{a_j} z)}{\prod_{j=1}^n (1 - \overline{b_j} z)} q$$

for some  $q \in \mathcal{P}_{n-m}$ . □

**Isometric Multipliers.** To finish this section we would like to make a connection with the so-called Aleksandrov-Clark measures [9, 46] (defined in (2.20)) in the setting of isometric multipliers. Recall that  $\mathcal{M}_i(u, v)$  is the set of isometric multipliers from  $\mathcal{K}_u$  to  $\mathcal{K}_v$  (see (3.6)). Using Aleksandrov's isometric embedding theorem [2] (see also [5, Theorem 1.1]), we obtain the following:

**Theorem 4.11.** *Suppose  $u$  and  $v$  are inner functions. Then the following are equivalent:*

- (i)  $\varphi \in \mathcal{M}_i(u, v)$ ;
- (ii)  $\varphi u \in \mathcal{K}_{zv}$  and there exist an  $\alpha \in \mathbb{T}$  and  $b \in H^\infty$ ,  $\|b\|_\infty \leq 1$ , such that  $|\varphi|^2 dm = \sigma_{bu}^\alpha$ , where  $\sigma_{bu}^\alpha$  is the Aleksandrov-Clark measure for the function  $bu$  at  $\alpha$ .

Observe how this requires that the Aleksandrov-Clark measure for the function  $bu$  to be absolutely continuous with respect to  $m$ .

Here is an example of an isometric multiplier stemming from work of Hitt [34] and Bourgain's factorization theorem [7]: Given two inner functions  $u$  and  $v$  with  $(\mathcal{K}_v \cap uH^2) \neq \{0\}$  (e.g.,  $u$  is a Blaschke product whose zeros are those of a function from  $\mathcal{K}_v \setminus \{0\}$ ), then

$$\frac{1}{u}(\mathcal{K}_v \cap uH^2) = \text{Ker } T_{\overline{v}u} = G\mathcal{K}_J$$

where  $G$  is an extremal function for  $\text{Ker } T_{\overline{v}u}$ , i.e., the unique solution to

$$\sup\{\Re f(0) : f \in \text{Ker } T_{\overline{v}u}, \|f\| \leq 1\}$$

and  $J$  is a suitable inner function satisfying  $J(0) = 0$ . Then  $Gu$  multiplies isometrically  $\mathcal{K}_J$  into  $\mathcal{K}_v$ , and we notice that  $|G|^2$  is the Radon-Nikodym derivative of an Aleksandrov-Clark measure associated with the function  $bu$ , where  $b$  can be explicitly expressed as a function depending on  $G$  [47]. We will develop this idea further in Section 8

## 5. SUB-LEVEL SETS

In this section we discuss some results using sub-level sets of inner functions. The first one uses the following ‘‘maximum principle’’ result of Cohn [13].

**Theorem 5.1** (Cohn). *Suppose  $\Theta$  is inner and  $f \in \mathcal{K}_\Theta$  is bounded on  $\{|\Theta| < \epsilon\}$  for some  $\epsilon \in (0, 1)$ . Then  $f \in H^\infty$ .*

This result can be used to show that under certain circumstances, all multipliers must be bounded.

**Corollary 5.2.** *Let  $u$  and  $v$  be inner. If, for some  $\epsilon_1, \epsilon_2 \in (0, 1)$ ,  $\{|v| < \epsilon_2\} \subseteq \{|u| < \epsilon_1\}$ , then  $\mathcal{M}(u, v) = \text{Ker } T_{\overline{v}u} \cap H^\infty$ .*

*Proof.* Let  $\varphi \in \mathcal{M}(u, v)$ . Recall the estimate (3.9) which says that

$$|\varphi(\lambda)|^2(1 - |u(\lambda)|^2) \lesssim (1 - |v(\lambda)|^2), \quad \lambda \in \mathbb{D}.$$

This says that for  $\lambda \in \{|v| < \epsilon_2\} \subseteq \{|u| < \epsilon_1\}$  we have

$$|\varphi(\lambda)|^2 \lesssim \frac{1}{1 - \epsilon_1^2}$$

and thus  $\varphi$  is bounded on  $\{|v| < \epsilon_2\}$ . Since  $k_0^u = 1 - \overline{u(0)}u \in \mathcal{K}_u$  (see (2.8)) and bounded on  $\mathbb{D}$ , we see that  $k_0^u\varphi \in \mathcal{K}_v$  and bounded on  $\{|v| < \epsilon_2\}$ . Applying Theorem 5.1 we get that  $k_0^u\varphi \in H^\infty$ . But since

$k_0^u$  is invertible in  $H^\infty$  we conclude that  $\varphi \in H^\infty$ . To finish, apply Corollary 4.2.  $\square$

**Example 5.3.** Let  $u$  be any singular inner function and  $v = u^\alpha$  for some  $\alpha > 1$  (or perhaps  $u$  a Blaschke product, or any inner function, and  $\alpha \in \mathbb{N}$ ). Notice that  $u$  divides  $v$  and so  $\mathcal{M}(u, v) \neq \{0\}$  (Proposition 3.11). Furthermore if  $\epsilon_2 \in (0, 1)$  and  $z \in \{|v| < \epsilon_2\}$  then

$$|u(z)|^{1/\alpha} \leq \epsilon_2^{1/\alpha}.$$

Setting  $\epsilon_1 = \epsilon_2^{1/\alpha}$  we see that  $\{|v| < \epsilon_2\} \subseteq \{|u| < \epsilon_1\}$ . Corollary 5.2 yields  $\{0\} \subsetneq \mathcal{M}(u, v) \subseteq H^\infty$ . Combine this with Corollary 4.4 to see that

$$\mathcal{M}_b(u, v) = \mathcal{M}(u, v) = \mathcal{K}_{zu^{\alpha-1}} \cap H^\infty.$$

Sub-level sets also play a role in describing the Carleson measures on model spaces. Here we use a theorem of Cohn [11, 12] which, in a specific situation, characterizes the Carleson measures for  $\mathcal{K}_\Theta$  (equivalently  $\mathcal{M}(\mathcal{K}_\Theta, H^2)$ ). To do this, we first need a definition.

**Definition 5.4.** We say that an inner function  $\Theta$  satisfies the *connected level set condition* if the sublevel set  $\{|\Theta| < \epsilon\}$  is connected for some  $\epsilon > 0$ .

For instance, any finite Blaschke product, the atomic inner function

$$u(z) = \exp\left(\frac{z+1}{z-1}\right),$$

and the Blaschke product with zeros  $\{1 - r^n\}_{n \geq 1}$ , where  $r \in (0, 1)$ , satisfy this connected level set condition.

**Theorem 5.5** (Cohn). *Suppose  $\Theta$  is a inner function satisfying the connected level set condition. Then  $f \in H^2$  is a multiplier from  $\mathcal{K}_\Theta$  to  $H^2$ , equivalently,  $|f|^2 dm$  is a Carleson measure for  $\mathcal{K}_\Theta$ , if and only if*

$$\sup_{\lambda \in \mathbb{D}} (1 - |\Theta(\lambda)|^2) \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} |f(\xi)|^2 dm(\xi) < \infty.$$

This allows us to restate the first part of Theorem 4.1 and Corollary 4.5 using the above, more tangible, condition for  $|f|^2 dm$  to be a Carleson measure for a model space.

**Theorem 5.6.** *Suppose  $u, v$  are inner and  $u$  satisfies the connected level set condition. Then the following are equivalent:*

$$(i) \ \varphi \in \mathcal{M}(u, v);$$

(ii)  $\varphi S^*u \in \mathcal{K}_v$  and

$$\sup_{\lambda \in \mathbb{D}} (1 - |u(\lambda)|^2) \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} |\varphi(\xi)|^2 dm(\xi) < \infty.$$

(iii)  $\varphi \in \text{Ker } T_{\overline{v}u}$  and

$$\sup_{\lambda \in \mathbb{D}} (1 - |u(\lambda)|^2) \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} |\varphi(\xi)|^2 dm(\xi) < \infty.$$

**Corollary 5.7.** *Suppose  $u$  and  $v$  are inner functions,  $u$  satisfies the connected level set condition, and  $v = uI$ . Then the following are equivalent:*

(i)  $\varphi \in \mathcal{M}(u, v)$ ;

(ii)  $\varphi \in \mathcal{K}_{zI}$  and

$$\sup_{\lambda \in \mathbb{D}} (1 - |u(\lambda)|^2) \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} |\varphi(\xi)|^2 dm(\xi) < \infty.$$

## 6. EXAMPLES INVOLVING FROSTMAN SHIFTS

In this short section, we give a few more examples of our main theorem.

For an inner function  $u$  and  $\lambda \in \mathbb{D}$ , recall from (1.3) the definition of the Frostman shift  $u_\lambda$  of  $u$ . Suppose that  $u_*$  is an inner factor of  $u_\lambda$  and

$$I := \frac{u_\lambda}{u_*}.$$

We want to classify  $\mathcal{M}(I, u)$  and  $\mathcal{M}_b(I, u)$ .

**Theorem 6.1.** *With the notation above, we have the following:*

(i) *If  $\dim(\mathcal{K}_{u_*}) < \infty$ , then*

$$\mathcal{M}(I, u) = \mathcal{M}_b(I, u) = (1 - \overline{\lambda}u)\mathcal{K}_{zu_*}.$$

(ii) *If  $\dim(\mathcal{K}_{u_*}) = \infty$ , then*

$$\mathcal{M}_b(I, u) = (1 - \overline{\lambda}u)\mathcal{K}_{zu_*} \cap H^\infty \subseteq \mathcal{M}(I, u) \subseteq (1 - \overline{\lambda}u)\mathcal{K}_{zu_*}.$$

*Proof.* Use the identity

$$\mathcal{M}(I, u) = \frac{1}{1 - \overline{\lambda}u} \mathcal{M}(I, u_\lambda)$$

and the observation that  $u_\lambda = Iu_*$ . □

**Example 6.2.** Applying Theorem 6.1 to  $\mathcal{M}(u_\lambda, u)$  we see that  $u_* \equiv 1$  and so

$$\mathcal{M}_b(u_\lambda, u) = \mathcal{M}(u_\lambda, u) = \mathbb{C}(1 - \bar{\lambda}u).$$

In other words, the multiplier space here is one dimensional.

Notice how this is Crofoot's theorem (see Theorem 3.14 above). With a little extra effort, we can obtain higher dimensional multiplier spaces as with the following example.

**Example 6.3.** Define

$$u^\lambda(z) = \frac{u(z) - u(\lambda)}{1 - \overline{u(\lambda)}u(z)} \frac{1 - \bar{\lambda}z}{z - \lambda} = \frac{u_{u(\lambda)}(z)}{b_\lambda(z)}, \quad \lambda \in \mathbb{D} \setminus \{0\},$$

where

$$b_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda}z}$$

is the single Blaschke factor. Apply Theorem 6.1 (with  $I = u^\lambda, u_* = b_\lambda, u_*I = u_{u(\lambda)}$ ) to see that

$$\begin{aligned} \mathcal{M}_b(u^\lambda, u) &= \mathcal{M}(u^\lambda, u) \\ &= (1 - \overline{u(\lambda)}u)\mathcal{K}_{zb_\lambda} \\ &= \bigvee \left\{ 1 - \overline{u(\lambda)}u, \frac{1 - \overline{u(\lambda)}u}{1 - \bar{\lambda}z} \right\}. \end{aligned}$$

In other words, the multiplier space here is two dimensional.

## 7. THE UPPER-HALF PLANE

Recall the definitions of the Hardy space  $\mathcal{H}^2$  of the upper-half plane from (2.30), the model space  $\mathcal{K}_U$  from (2.34), and the natural unitary operator  $\mathcal{U} : H^2 \rightarrow \mathcal{H}^2$  from (2.32). Given inner functions  $u, v$  on  $\mathbb{D}$ , we set  $U = u \circ \omega$  and  $V = v \circ \omega$ , where  $\omega : \mathbb{C}_+ \rightarrow \mathbb{D}$  is the Möbius transform from (2.33). These functions  $U$  and  $V$  will be inner on  $\mathbb{C}_+$ .

**Multipliers and kernels.** In this section we need the elementary Blaschke factor on  $\mathbb{C}_+$  with zero at  $i$ :

$$(7.1) \quad b_i^+(z) := \frac{z - i}{z + i}.$$

We begin with some elementary but useful facts that will be needed in our later discussions.

**Lemma 7.2.** *Let  $\psi \in L^\infty(\mathbb{T})$  and  $\Psi = \psi \circ \omega$ . Then*

$$f \in \text{Ker } T_\psi \iff F := \mathcal{U}f \in \text{Ker } T_\Psi.$$

*Proof.* If  $f \in \text{Ker } T_\psi$ , then  $f\psi \perp_{H^2} H^2$  and so  $\mathcal{U}(f\psi) \perp_{\mathcal{H}^2} \mathcal{H}^2$ . Moreover,

$$\begin{aligned} \mathcal{U}(f\psi)(w) &= \frac{1}{w+i} (f\psi) \circ \omega \\ &= \frac{1}{w+i} (f \circ \omega) \times (\psi \circ \omega) \\ &= \mathcal{U}f \times \Psi, \end{aligned}$$

and hence  $F = \mathcal{U}f \in \text{Ker } T_\Psi$ . The proof of the converse is similar.  $\square$

**Lemma 7.3.**  *$\varphi \in \mathcal{M}(u, v)$  if and only if  $\Phi = \varphi \circ \omega \in \mathcal{M}(U, V)$ .*

*Proof.* Recalling that  $\mathcal{U}\mathcal{K}_u = \mathcal{K}_U$  and  $\mathcal{U}\mathcal{K}_v = \mathcal{K}_V$  from (2.36), we note that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{K}_u & \xrightarrow{\varphi} & \mathcal{K}_v \\ \mathcal{U} \downarrow & & \downarrow \mathcal{U} \\ \mathcal{K}_U & \xrightarrow{\Phi} & \mathcal{K}_V \end{array}$$

Indeed, pick  $f \in \mathcal{K}_U$ , then

$$\begin{aligned} (\mathcal{U}\varphi\mathcal{U}^{-1}f)(z) &= \frac{1}{z+i} \varphi\left(\frac{z-i}{z+i}\right) \times (\mathcal{U}^{-1}f)\left(\frac{z-i}{z+i}\right) \\ &= \frac{1}{z+i} \varphi\left(\frac{z-i}{z+i}\right) \frac{2i}{1 - \frac{z-i}{z+i}} f(z) \\ &= \Phi(z)f(z). \end{aligned} \quad \square$$

Thus kernels are transformed via the operator  $\mathcal{U}$  and multipliers are transformed via composition with  $\omega$ .

Let us set

$$k_i(z) = \frac{1}{z+i}$$

and observe that this is the reproducing kernel for  $\mathcal{H}^2$  at  $i$  (up to a multiplicative constant). In particular,

$$\mathcal{U}f = k_i \times (f \circ \omega), \quad f \in H^2.$$

**Corollary 7.4.** *With the notation from (7.1), the following are equivalent for  $\Phi$  analytic on  $\mathbb{C}_+$ :*

$$(i) \quad \Phi \in \mathcal{M}(U, V);$$

$$(ii) \quad \Phi k_i \in \text{Ker } T_{b_i^+ V U} \text{ and } |\Phi|^2 dx \text{ is a Carleson measure for } \mathcal{K}_U.$$

*Proof.* (i)  $\implies$  (ii): Suppose that  $\Phi \in \mathcal{M}(U, V)$ . By Lemma 7.3,  $\Phi = \varphi \circ \omega$  for some  $\varphi \in \mathcal{M}(u, v)$  and hence  $\varphi \in \text{Ker } T_{\overline{v}u}$ . Then

$$\mathcal{U}\varphi = k_i \times (\varphi \circ \omega) \in \text{Ker } T_{b_i^+ V U}$$

(Lemma 7.2). The Carleson measure condition is immediate.

(ii)  $\implies$  (i): By assumption we have  $\Phi k_i = \mathcal{U}\varphi$  for some  $\varphi \in \text{Ker } T_{\overline{v}u}$ . By the Carleson measure condition we also have  $F\Phi \in L^2(\mathbb{R})$  for every  $F \in \mathcal{K}_U$ . Then  $\mathcal{U}^{-1}(F\Phi) \in L^2(\mathbb{T})$ . Observe that

$$\mathcal{U}^{-1}(F\Phi) = (\mathcal{U}^{-1}F) \times \varphi.$$

Since  $\mathcal{U}^{-1}F$  runs through the  $\mathcal{K}_u$ -functions when  $F$  runs through the  $\mathcal{K}_U$  functions, we see that  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ . As a result,  $\varphi \in \text{Ker } T_{\overline{v}u}$  and  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ . Hence by Theorem 4.1,  $\varphi \in \mathcal{M}(u, v)$ , and by Lemma 7.3,  $\Phi = \varphi \circ \omega \in \mathcal{M}(U, V)$ .  $\square$

In Example 4.7 we gave an example of when  $\mathcal{M}(u, v) \subsetneq \text{Ker } T_{\overline{v}u}$ . We now discuss a situation when the Carleson condition becomes more tractable. In the situation we have in mind, we will see that the multiplier space contains an associated Toeplitz kernel. We need the notion of sub-level set of an inner function  $U$  for  $\mathbb{C}_+$ :

$$L(U, \epsilon) = \{z \in \mathbb{C}_+ : |U(z)| < \epsilon\}, \quad \epsilon \in (0, 1).$$

We will also define

$$(7.5) \quad \mathbb{C}_\eta = \{z \in \mathbb{C} : \Im z > \eta\}, \quad \eta > 0.$$

**Lemma 7.6.** *Let  $U$  be an inner function in  $\mathbb{C}_+$  such that  $L(U, \epsilon) \subseteq \mathbb{C}_\eta$  for some  $\epsilon \in (0, 1)$  and  $\eta > 0$ . Let  $\mu$  be a positive Borel measure on  $\mathbb{R}$ . The following assertions are equivalent:*

$$(i) \quad \mu \text{ is a Carleson measure for } \mathcal{K}_U.$$

$$(ii) \quad \text{We have}$$

$$M := \sup_{x \in \mathbb{R}} \mu([x, x + \eta]) < \infty.$$

Note that instead of integrating over intervals of length  $\eta$  in the definition of  $M$ , we could, for instance, also integrate over intervals of length 1.

Inner functions  $U$  for which the level set is bounded away from the real line are precisely given by meromorphic inner  $U$  (inner on  $\mathbb{C}_+$  with an analytic continuation to a neighborhood of the closure of  $\mathbb{C}_+$ ) for which  $U' \in L^\infty(\mathbb{R})$  [20].

*Proof of Lemma 7.6.* (ii)  $\implies$  (i): using a result by Treil and Volberg [53], it suffices to check the Carleson measure condition for those Carleson boxes

$$S(I) = \{z = x + iy \in \mathbb{C}_+ : x \in I, y < |I|\}, \quad I \subseteq \mathbb{R},$$

where  $I$  is an interval, and  $|I|$  is its length, which meet the level set  $L(U, \epsilon)$ . By assumption  $L(U, \epsilon) \subseteq \mathbb{C}_\eta$  so that we only need to test the intervals  $I$  for which  $|I| \geq \eta$ . Then  $|I| = k\eta + r$ , where  $k \in \mathbb{N}$  and  $r \in [0, \eta)$ . Setting  $I = [a, b]$  we get

$$\begin{aligned} \mu(I) &\leq \sum_{j=0}^k \mu([a + j\eta, a + (j+1)\eta]) \\ &\leq (k+1)M = \frac{(k+1)M}{k\eta} k\eta \\ &\leq \frac{(k+1)M}{k\eta} |I| \leq 2 \frac{M}{\eta} |I|. \end{aligned}$$

(i)  $\implies$  (ii): since  $U$  is a meromorphic inner function, the kernel

$$k_{x_0}^U(z) = \frac{1 - \overline{U(x_0)}U(z)}{z - x_0}$$

belongs to  $\mathcal{H}_U$  and  $\|k_{x_0}^U\|_2^2 = |U'(x_0)|$  for every  $x_0 \in \mathbb{R}$ . Using the fact that  $\mu$  is a Carleson measure for  $\mathcal{H}_U$ , there exists a constant  $C > 0$  such that, for every  $x_0 \in \mathbb{R}$ , we have

$$(7.7) \quad \int_{\mathbb{R}} |k_{x_0}^U(t)|^2 d\mu(t) \leq C |U'(x_0)|.$$

Write now  $U(t) = e^{i\varphi(t)}$ , where  $\varphi$  is an increasing branch of the argument of  $U$ . Note that since  $U$  is a meromorphic inner function,  $\varphi$  is real-analytic. Then we get

$$(7.8) \quad |k_{x_0}^U(t)| = \frac{|U(t) - U(x_0)|}{|t - x_0|} = 2 \frac{\left| \sin \left( \frac{\varphi(t) - \varphi(x_0)}{2} \right) \right|}{|t - x_0|}.$$



Let

$$\delta = \min\left(\frac{\pi}{\|U'\|_\infty}, \frac{\eta}{2}\right).$$

If  $|t - x_0| \leq \delta$ , we have

$$|\varphi(t) - \varphi(x_0)| \leq \|U'\|_\infty |t - x_0| \leq \pi.$$

Using the inequality of convexity  $\sin(u) \geq \frac{2}{\pi}u$  for  $u \in (0, \pi/2)$ , we deduce that

$$(7.9) \quad \left| \sin\left(\frac{\varphi(t) - \varphi(x_0)}{2}\right) \right| \geq \frac{|\varphi(t) - \varphi(x_0)|}{\pi}.$$

By the mean value theorem, there is a point  $c \in (x_0 - \delta, x_0 + \delta)$  such that

$$|\varphi(t) - \varphi(x_0)| = |t - x_0| |\varphi'(c)| = |t - x_0| |U'(c)|.$$

Combining this last equation with (7.8) and (7.9) gives

$$(7.10) \quad |k_{x_0}^U(t)| \geq 2 \frac{|U'(c)|}{\pi}.$$

Remembering that  $U$  is a meromorphic inner function with  $L(U, \varepsilon) \subseteq \mathbb{C}_\eta$ , we have  $U(z) = e^{iaz} B(z)$  where  $B$  is a Blaschke product associated to a Blaschke sequences of points  $(z_n)_n \subseteq \mathbb{C}_\eta$  and  $a > 0$  (because  $U$  is assumed to have a non constant singular inner factor). Then it is well known that

$$|U'(t)| = a + 2 \sum_n \frac{\Im z_n}{|t - z_n|^2}, \quad t \in \mathbb{R}.$$

Since

$$\frac{|x_0 - z_n|}{|c - z_n|} \geq 1 - \frac{|x_0 - c|}{|c - z_n|} \geq 1 - \frac{\delta}{\eta} \geq \frac{1}{2},$$

we get that

$$|U'(c)| = a + 2 \sum_{n \geq 1} \frac{\Im z_n}{|c - z_n|^2} \geq a + \frac{1}{2} \sum_{n \geq 1} \frac{\Im z_n}{|x_0 - z_n|^2} \geq \frac{1}{4} |U'(x_0)|,$$

which, using (7.10), yields

$$|k_{x_0}^U(t)| \geq \frac{1}{2\pi} |U'(x_0)|,$$

for every  $t$  such that  $|t - x_0| \leq \delta$ . We thus get from (7.7)

$$\mu([x_0, x_0 + \delta]) \leq \frac{C(2\pi)^2}{|U'(x_0)|^2} \leq \frac{4C\pi^2}{a^2}.$$

Now let  $k \in \mathbb{N}$  be the smallest integer such that  $k\delta > \eta$ . If we iterate the last estimate we obtain that

$$\mu([x_0, x_0 + \eta]) \leq \frac{4C\pi^2 k}{a^2}.$$

Since this is true for every  $x_0 \in \mathbb{R}$ , we get that

$$M \leq \frac{4C\pi^2 k}{a^2} < \infty. \quad \square$$

**Theorem 7.11.** *Let  $U$  and  $V$  be inner functions with  $U(i) = 0$ . Suppose  $L(U, \epsilon) \subseteq \mathbb{C}_\eta$  for some  $\epsilon \in (0, 1)$  and  $\eta > 0$ . Then*

$$\mathcal{M}(U, V) = \left\{ \Phi \in (z + i) \operatorname{Ker} T_{b_i^+ V U}^- : M := \sup_{x \in \mathbb{R}} \int_x^{x+\eta} |\Phi(t)|^2 dt < \infty \right\}.$$

*Proof.* Observe that

$$\Phi k_i \in \operatorname{Ker} T_{b_i^+ V U}^- \iff \Phi \in (z + i) \operatorname{Ker} T_{b_i^+ V U}^-.$$

It remains to apply Corollary 7.4 and Lemma 7.6.  $\square$

Observe that when  $\Phi \in (z + i) \operatorname{Ker} T_{b_i^+ V U}^-$  and  $\Phi \in \mathcal{H}^2 \cup \mathcal{H}^\infty$ , then we clearly have

$$\sup_{x \in \mathbb{R}} \int_x^{x+\eta} |\Phi(x)|^2 dx < \infty.$$

Let us consider the case when the multiplier is in  $\mathcal{H}^2$ , in other words we are interested when

$$\Phi \in \left( (z + i) \operatorname{Ker} T_{b_i^+ V U}^- \right) \cap \mathcal{H}^2.$$

As it turns out, for this it suffices to add just one zero to the inner function defining the initial space  $\mathcal{K}_U$ .

**Lemma 7.12.** *We have*

$$F \in \operatorname{Ker} T_{U \bar{V}} \iff F \in \left( (z + i) \operatorname{Ker} T_{b_i^+ V U}^- \right) \cap \mathcal{H}^2.$$

*Proof.* The function  $F$  belongs to  $\operatorname{Ker} T_{U \bar{V}}$  if and only if there is a  $\psi \in \mathcal{H}^2$  such that

$$U \bar{V} F = \bar{\psi}.$$

This implies that on  $\mathbb{R}$  we have

$$\begin{aligned} U(x) \overline{V(x) b_i^+(x)} F(x) k_i(x) &= U(x) \overline{V(x) \left( \frac{x-i}{x+i} \right)} F(x) \frac{1}{x+i} \\ &= \overline{\psi(x) \left( \frac{x-i}{x+i} \right)} \frac{1}{x+i} \\ &= \overline{\psi(x) \left( \frac{x+i}{x-i} \right)} \frac{1}{x+i} \end{aligned}$$

$$\begin{aligned}
&= \overline{\psi(x)} \frac{1}{x+i} \\
&= \overline{(\psi k_i)(x)}
\end{aligned}$$

Hence  $Fk_i \in \text{Ker } T_{b_i^+ V U}$  and so  $F \in (z+i) \text{Ker } T_{b_i^+ V U}$ .

The converse argument is in the same spirit. Indeed, when

$$F \in (z+i) \text{Ker } T_{b_i^+ V U} \cap \mathcal{H}^2,$$

we get

$$F(x)(U(x)\overline{V(x)}) = \overline{\psi(x)}(x-i) = \overline{\psi(x)(x+i)}.$$

Since  $F \in \mathcal{H}^2$ , and  $U\overline{V}$  is bounded, we deduce that  $\psi(z+i) \in \mathcal{H}^2$ , and so  $F \in \text{Ker } T_{U\overline{V}}$ .  $\square$

One can see from the discussion above that when  $\text{Ker } T_{U\overline{V}} \neq \{0\}$ , every  $F \in \text{Ker } T_{U\overline{V}}$  produces a multiplier  $F \in \mathcal{M}(U, V)$ .

Let us state this observation in the following separate result which, in this particular situation, characterizes those multipliers which belong to  $\mathcal{H}^2$  in terms of kernels of Toeplitz operators.

**Corollary 7.13.** *Let  $U$  and  $V$  be inner functions with  $U(i) = 0$ . Suppose  $L(U, \epsilon) \subseteq \mathbb{C}_\eta$  for some  $\epsilon \in (0, 1)$  and  $\eta > 0$ . Then*

$$\mathcal{M}(U, V) \cap \mathcal{H}^2 = \text{Ker } T_{U\overline{V}}.$$

**Bounded multipliers.** In [8] and [19], the authors determine when a model space in the upper-half plane is contained in  $\mathcal{H}^\infty$ . In particular, Dyakonov proved that the following conditions are equivalent:

- (i)  $U' \in L^\infty(\mathbb{R})$ .
- (ii) For some  $\eta$  and some  $\varepsilon > 0$ ,  $L(U, \varepsilon) \subseteq \mathbb{C}_\eta$ .
- (iii)  $\mathcal{H}_U \subseteq \mathcal{H}^\infty$ .

**Remark 7.14.** In the disk case, a well-known fact states that a model space is contained in  $H^\infty$  if and only if the corresponding inner function is a finite Blaschke product (see e.g., [8, Thm. 4.2]). This also follows from a theorem by Grothendieck [45, p. 117] which shows that a closed subspace of  $L^2$  which is contained in  $L^\infty$  is necessarily finite dimensional.

In Example 5.3, we construct two inner functions  $u_1$  and  $v$  such that all the multipliers from  $K_{u_1}$  into  $K_v$  are bounded and in that example,

the inner function  $u_1$  divides  $v$ . We now construct an example without any divisibility condition.

**Example 7.15.** Let

$$u(z) = \exp\left(\frac{z+1}{z-1}\right)$$

and  $U = u \circ \omega$ . Then the corresponding level set for  $U$  in  $\mathbb{C}_+$ , is  $\mathbb{C}_\eta$  (see (7.5)) for some  $\eta > 0$ . Now pick a separated sequence  $\{x_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R}$ , i.e.,

$$|x_n - x_m| \geq \delta > 0, \quad m, n \in \mathbb{Z}, m \neq n,$$

and let  $V$  be the Blaschke product (in the upper-half plane) whose zeros are precisely  $\{x_n + i\}_{n \in \mathbb{Z}}$ . Then  $V$  is an interpolating Blaschke product whose level set is contained in some  $\mathbb{C}_{\eta'}$ ,  $\eta' > 0$  (it is a union of Euclidean disks centered about the points  $x_n + i$ ). If  $v = V \circ \omega^{-1}$ , is the corresponding Blaschke product in  $\mathbb{D}$ , then, for some  $\epsilon_1, \epsilon_2 \in (0, 1)$ , the sub-level set inclusion

$$\{|v| < \epsilon_2\} \subseteq \{|u| < \epsilon_1\}$$

is satisfied (note that the zeros of  $v$  lie in an oricycle) and hence by Corollary 5.2 we have  $\mathcal{M}(u, v) \subseteq H^\infty$ . At this point there is no guarantee that  $\mathcal{M}(u, v)$  is non-zero. To guarantee that  $\mathcal{M}(u, v)$  is non-zero, we use Theorem 4.1 and results from [35, 39] to obtain sufficient conditions for  $\text{Ker } T_{\overline{v}u} \neq \{0\}$ . For example, one can arrange this by taking a sufficiently dense separated sequence  $\{x_n\}_{n \in \mathbb{Z}}$ .

**Multipliers and Ahern-Clark points.** When  $\mathcal{M}(u, v) \neq \{0\}$  we know from Proposition 3.30 that  $\sigma(u) \subseteq \sigma(v)$ . Is it the case that the Ahern-Clark points for  $u$  are also Ahern-Clark points for  $v$ ? Recall the Ahern-Clark property from (2.17): every  $f \in \mathcal{K}_u$  has a non-tangential limit at  $\zeta \in \mathbb{T}$  if and only if  $u$  has a finite angular derivative at  $\zeta$ . In the upper-half plane case note that  $\infty$  is an Ahern-Clark point for a model space  $\mathcal{H}_U$  precisely when  $U \circ \omega^{-1}$  has a finite angular derivative at  $z = 1$  (equivalently  $U$  has an angular derivative at  $\infty$ ).

**Proposition 7.16.** *There exists two inner functions  $U$  and  $V$  in the upper half plane such that  $\mathcal{M}(U, V)$  is non trivial,  $\sigma(U) = \sigma(V) = \{\infty\}$ , and  $V$  has an angular derivative at  $\infty$  while  $U$  does not.*

In this situation, the multiplier  $\varphi$  must tend non-tangentially to 0 when  $z \rightarrow \infty$  in a Stolz angle. This is because any function  $\varphi F \in \mathcal{H}_V$  ( $F \in \mathcal{H}_U$ ) admits non-tangential limits while  $F$ , in general, does not.

*Proof.* For  $n \geq 1$  let  $\mu_n = i2^n$  and  $\lambda_n = n + \frac{i}{n^2}$ . We define the inner functions  $U = B_{\Lambda_1}$  and  $V = B_{\Lambda_2}$  where  $\Lambda_1 = \{\mu_n\}_{n \geq 1}$  and  $\Lambda_2 = \{\lambda_n\}_{n \geq 1}$ . The boundary spectrum of both functions  $U$  and  $V$  is simply the sole point  $\{\infty\}$ . It is known that  $\infty$  is an Ahern-Clark point for a Blaschke product  $B_\Lambda$ , where  $\Lambda = \{\nu_n\}_{n \geq 1}$ , if and only if  $\sum_{n \geq 1} \Im \nu_n < \infty$ . From this the assertions on the Ahern-Clark properties for  $U$  and  $V$  follow immediately.

It remains to check that  $\text{Ker } T_{\overline{b_i^+ V U}}$  is non-trivial. To do this we will use a result by Makarov and Poltoratski [39, Corollary of Section 4.1] which says that if  $\gamma$ , defined by  $\overline{b_i^+ V U} = e^{i\gamma}$  on  $\mathbb{R}$ , can be written as a sum of a bounded and a decreasing function then, possibly adding  $n$  zeros to  $V$ , we can obtain

$$\text{Ker } T_{\overline{UV(b_i^+)^{n+1}}} \cap H^\infty(\mathbb{C}_+) \neq \{0\}.$$

Here  $V$  has to be assumed tempered, which means that  $V'$  has at most polynomial growth. Let us now check that we are in the situation of the Marakov-Poltoratski result cited above.

Recall that for every  $B_\Lambda(x) = e^{i\gamma(x)}$ ,  $x \in \mathbb{R}$ ,  $\Lambda = \{\nu_n\}$ , we have

$$\gamma'(x) = \sum_{n \geq 1} \frac{d}{dx} \log \frac{x - \nu_n}{x - \bar{\nu}_n} = \sum_{n \geq 1} \frac{2\Im \nu_n}{|x - \nu_n|^2}$$

Apply this to  $U = B_{\Lambda_1}(x) = e^{i\gamma_1(x)}$ ,  $x \in \mathbb{R}$ ,

$$\begin{aligned} \gamma_1'(x) &= \sum_{n \geq 1} \frac{2\Im \mu_n}{|x - \mu_n|^2} = \sum_{n \geq 1} \frac{2 \times 2^n}{|x - i2^n|^2} = \sum_{n \geq 1} \frac{2^{n+1}}{x^2 + 2^{2n}} \\ &\leq \sum_{x \leq 2^n} \frac{2^{n+1}}{x^2 + 2^{2n}} + \sum_{x > 2^n} \frac{2^{n+1}}{x^2 + 2^{2n}} \\ &\leq \sum_{x \leq 2^n} \frac{2^{n+1}}{2^{2n}} + \sum_{x > 2^n} \frac{2^{n+1}}{x^2} \\ &\leq \sum_{x \leq 2^n} 2^{1-n} + \frac{1}{x^2} \sum_{x > 2^n} 2^{n+1} \\ &\lesssim \frac{2}{x} + \frac{2x}{x^2} = \frac{4}{x}. \end{aligned}$$

So that the argument of  $U$  grows at most logarithmically.

Analogously for  $V$ ,

$$\gamma_2'(x) = \sum_{n \geq 1} \frac{2\Im \lambda_n}{|x - \lambda_n|^2} = \sum_{n \geq 1} \frac{2/n^2}{|x - (n + i/n^2)|^2}$$

$$\begin{aligned}
(7.17) \quad &= \sum_{n \geq 1} \frac{2/n^2}{(x-n)^2 + 1/n^4} \\
&\geq \frac{2}{n_0^2(x-n_0)^2 + 1/n_0^2},
\end{aligned}$$

where we choose  $n_0$  such that  $x \in [n_0 - 1/2, n_0 + 1/2)$ . Let us estimate from below the increment of  $\gamma_2$  on  $[n_0 - 1/2, n_0 + 1/2)$ :

$$\begin{aligned}
&\gamma_2(n_0 + 1/2) - \gamma_2(n_0 - 1/2) \\
&= \int_{n_0-1/2}^{n_0+1/2} \gamma_2'(x) dx \geq \int_{n_0-1/2}^{n_0+1/2} \frac{2}{n_0^2(x-n_0)^2 + 1/n_0^2} dx \\
&= \int_{n_0-1/2}^{n_0+1/2} \frac{2n_0^2}{1 + (n_0^2x - n_0^3)^2} dx
\end{aligned}$$

Setting  $y = xn_0^2 - n_0^3$  gives  $dy = n_0^2 dx$  and translates the interval of integration to

$$[(n_0 - 1/2)n_0^2 - n_0^3, (n_0 + 1/2)n_0^2 - n_0^3] = [-n_0^2/2, n_0^2/2).$$

Hence

$$\gamma_2(n_0 + 1/2) - \gamma_2(n_0 - 1/2) \geq \int_{-n_0^2/2}^{n_0^2/2} \frac{2}{1+y^2} dy \rightarrow 2\pi, \quad n_0 \rightarrow \infty.$$

We deduce that  $\gamma_2$  increases at least linearly. This implies that the at most logarithmic growth of  $\gamma_1$  can be absorbed into the, at least, linear decrease of  $-\gamma_2$ , so that  $\gamma_1 - \gamma_2$  can be written as a decreasing function and a bounded function. From (7.17) it can also be deduced that the worst growth of  $\gamma_2'(x)$  is for  $x = n_0$ , then the term in  $n = n_0$  contributes with  $n_0^2 = x^2$  to the sum, while the remaining sum is essentially bounded by  $1/n_0 = 1/x$  which is neglectible. Hence the argument of  $V$  has at most quadratic growth so that  $V$  is tempered, and the result of Makarov and Poltoratski applies (In our case the  $n$  appearing in that theorem can be taken to be equal to 2).  $\square$

## 8. UNBOUNDED MULTIPLIERS

It turns out that unbounded multipliers between model spaces exist in abundance and can be created in various ways. In this section, we give three different examples of unbounded multipliers, each connecting with a different aspect of model spaces.

**Using analytic continuation.** Recall, again, the definition of the boundary spectrum  $\sigma(u)$  of an inner function  $u$  from (2.14).

**Theorem 8.1.** *Let  $u$  and  $v$  be inner functions,  $u(0) = 0$ , and  $v = uI$  for some inner function  $I$ . Suppose further that  $\sigma(u) \cap \sigma(I) = \emptyset$ . Then  $\mathcal{M}(u, v) = \mathcal{K}_{zI}$ . Furthermore, if  $I$  is not a finite Blaschke product then  $\mathcal{M}(u, v)$  contains unbounded functions.*

*Proof.* By Corollary 4.5, we just need to check  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$  for every  $\varphi \in \mathcal{K}_{zI}$ .

Let  $V$  be a two dimensional neighborhood of  $\sigma(I)$  that is far from  $\sigma(u)$ . By (2.15)  $\varphi$  extends analytically outside  $V$  and thus can be assumed to be bounded outside  $V$ . Similarly, every  $f \in \mathcal{K}_u$  extends analytically to  $V$  and can be assumed to be bounded there. Thus

$$\begin{aligned} \int_{\mathbb{T}} |\varphi f|^2 dm &= \int_{\mathbb{T} \setminus V} |\varphi f|^2 dm + \int_{\mathbb{T} \cap V} |\varphi f|^2 dm \\ &\lesssim \int_{\mathbb{T} \setminus V} |f|^2 dm + \int_{\mathbb{T} \cap V} |\varphi|^2 dm < \infty. \end{aligned}$$

By the Closed Graph Theorem,  $\varphi \in \mathcal{M}(\mathcal{K}_u, H^2)$ , equivalently,  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$ .

For the last part of the theorem, note that if  $I$  is not a finite Blaschke product then  $\mathcal{K}_I \subseteq \mathcal{K}_{zI}$  is infinite dimensional (see (2.18)) and thus contains an unbounded function (see Remark 7.14).  $\square$

It is interesting that one can have an unbounded multiplier from a finite dimensional model space to an infinite dimensional one. Of course, one can come up with plenty of examples of two infinite degree inner functions  $u$  and  $v$  satisfying the hypothesis of Theorem 8.1.

**An example using Bourgain factorization.** The previous example of  $\mathcal{M}_b(u, v) \subsetneq \mathcal{M}(u, v)$  relied on the fact that  $u$  was a divisor of  $v$ . Here we construct an unbounded multiplier based on the isometric multipliers discussed after Theorem 4.11 and Bourgain factorization.

**Proposition 8.2.** *There are inner functions  $u$  and  $v$ , with  $v$  not a multiple of  $u$ , such that  $\mathcal{M}_b(u, v) \subsetneq \mathcal{M}(u, v)$ .*

*Proof.* From Hayashi [33] (see also [47]) observe that if  $a \in H^\infty$ ,  $\|a\|_\infty \leq 1$ , and outer, and  $b$  is its Pythagorean mate (i.e.,  $b \in H^\infty$  and  $|a|^2 +$

$|b|^2 = 1$  a.e. on  $\mathbb{T}$ ), and such that  $G_0 := a/(1 - b)$  is rigid, then for every inner  $J$ , the function

$$G := \frac{a}{1 - Jb}$$

is an isometric multiplier on  $K_J$  to  $H^2$  and the space  $GK_J$  is the kernel of a Toeplitz operator. It is not very difficult to construct  $a$  and  $b$  such that  $G_0$ , and hence  $G$ , are unbounded. See [31] for specific examples of this.

Now from Bourgain's factorization [21, Theorem 1] we know that there is a triple  $(u, v, g)$  where  $u$  and  $v$  are inner functions (in fact Blaschke products),  $g$  is an invertible element of  $H^\infty$ , such that

$$(8.3) \quad GK_J = \frac{g}{u}(\mathcal{K}_v \cap uH^2).$$

The identity (8.3) tells us that  $\varphi := Gu/g \in \mathcal{M}(J, v)$ . Moreover,  $\varphi$  is unbounded since  $G$  is unbounded and  $g$  is an invertible element of  $H^\infty$ . If the inner function  $J$  is chosen so that its singular inner part (from (2.4)) is not constant, then, since  $v$  is a Blaschke product, the function  $v$  is not a multiple of  $J$ .  $\square$

**Resolving a question of Crofoot.** The previous two examples created unbounded multipliers, i.e.,  $\varphi\mathcal{K}_u \subseteq \mathcal{K}_v$ . However, there was no way to check that these multipliers were onto, i.e.,  $\varphi\mathcal{K}_u = \mathcal{K}_v$ . This next example, using de Branges spaces of entire functions, yields such an example - and hence resolving a long standing question of Crofoot [14, p. 244].

**Theorem 8.4.** *There are two inner functions  $u$  and  $v$  on  $\mathbb{D}$  and an unbounded analytic function  $\varphi$  such that  $\varphi\mathcal{K}_u = \mathcal{K}_v$ .*

We will prove the corresponding result for the model spaces  $\mathcal{K}_\Theta$  of the upper-half plane and then use Lemma 7.3. The construction is based on the relationship between the model subspaces generated by meromorphic inner functions and the de Branges spaces of entire functions [15].

First we define the Paley-Wiener class

$$\begin{aligned} PW &:= \{F \in \mathcal{O}(\mathbb{C}) : F|_{\mathbb{R}} \in L^2(\mathbb{R}), |F(z)| \leq C_F e^{\pi|\Im z|}\} \\ &\cong \{f \in L^2(\mathbb{R}) : \text{supp}(\mathcal{F}f) \subseteq [-\pi, \pi]\} \\ &\cong \left\{ F \in \mathcal{O}(\mathbb{C}) : \frac{F}{e^{-i\pi z}}, \frac{F^*}{e^{-i\pi z}} \in \mathcal{H}^2 \right\}, \end{aligned}$$



where

$$F^*(z) := \overline{F(\bar{z})}.$$

Next we need the de Branges spaces of entire functions. Let  $E$  be an entire function which belongs to the *Hermite–Biehler class*  $HB$ , that is to say,

$$|E(z)| \geq |E(\bar{z})|, \quad \Im z > 0$$

and  $E$  does not have any zeros in  $\mathbb{C}_+^-$  (the closed upper half plane). With  $E \in HB$ , define the *de Branges space*

$$\mathcal{H}(E) := \left\{ F \in \mathcal{O}(\mathbb{C}) : \frac{F}{E}, \frac{F^*}{E} \in \mathcal{H}^2 \right\}.$$

The norm in  $\mathcal{H}(E)$  is defined by

$$\|F\|_E = \left\| \frac{F}{E} \right\|_{L^2(\mathbb{R})}, \quad F \in \mathcal{H}(E).$$

If  $E \in HB$ , then  $\Theta = E^*/E$  is a meromorphic inner function in  $\mathbb{C}_+$ , meaning that  $\Theta$  is an inner function and that  $\Theta$  has an analytic continuation to an open neighborhood of  $\mathbb{C}_+^-$ . Conversely, each meromorphic inner function  $\Theta$  admits a representation  $\Theta = E^*/E$  for some entire function  $E \in HB$ . One can see from (2.35) that when  $\Theta = E^*/E$ , the operator  $F \mapsto F/E$  is unitary from  $\mathcal{H}(E)$  onto the model space  $\mathcal{H}_\Theta$ , that is to say,

$$(8.5) \quad \mathcal{H}_\Theta = \frac{1}{E} \mathcal{H}(E).$$

When  $E(z) = e^{-i\pi z}$ , one can check that  $E \in HB$ ,  $\Theta = E^*/E$  satisfies  $\Theta(z) = e^{2i\pi z}$ , and

$$\mathcal{H}_\Theta = e^{i\pi z} \mathcal{H}(E) = e^{i\pi z} PW,$$

where  $PW$  is the Paley-Wiener space defined earlier.

*Proof of Theorem 8.4.* Fix  $\delta \in (0, \frac{1}{4})$  and set

$$E_\delta(z) = (z+i) \prod_{k=1}^{\infty} \left( 1 - \frac{z}{k - \delta - ik^{-4\delta}} \right) \left( 1 - \frac{z}{-k + \delta - ik^{-4\delta}} \right).$$

It is shown in [38] that  $E_\delta \in HB$ ,

$$(8.6) \quad \mathcal{H}(E_\delta) = PW,$$

with equivalent norms, and

$$(8.7) \quad |E_\delta(x)| \simeq (1 + |x|)^{2\delta} \text{dist}(x, \Lambda_\delta), \quad x \in \mathbb{R},$$

where

$$\begin{aligned}\Lambda_\delta &= E_\delta^{-1}(\{0\}) \\ &= \{k - \delta - ik^{-4\delta} : k \geq 1\} \cup \{-k + \delta - ik^{-4\delta} : k \geq 1\} \cup \{-i\}.\end{aligned}$$

If we define  $I_\delta = E_\delta^*/E_\delta$ , then  $I_\delta$  is a meromorphic inner function on  $\mathbb{C}_+$ . Define  $\varphi_\delta(z) = e^{i\pi z} E_\delta(z)$  and use (8.5) and (8.6) to obtain

$$\varphi_\delta \mathcal{K}_{I_\delta} = e^{i\pi z} E_\delta \mathcal{K}_{I_\delta} = e^{i\pi z} \mathcal{H}(E_\delta) = e^{i\pi z} PW = \mathcal{K}_\Theta,$$

where  $\Theta(z) = e^{2\pi iz}$ . Hence  $\varphi_\delta$  is a multiplier from  $\mathcal{K}_{I_\delta}$  onto  $\mathcal{K}_\Theta$ . We now argue that  $\varphi_\delta$  is unbounded. Indeed, the zero set  $\Lambda_\delta$  of  $E_\delta$  contains the zeros

$$z_k = (k - \delta) - ik^{-4\delta}, \quad k \geq 1.$$

For each interval  $(k - \delta, k + 1 - \delta)$ , the zeros  $z_k$  and  $z_{k+1}$  lie just below the respective endpoints  $k - \delta$  and  $k + 1 - \delta$ . If

$$x_k := \frac{2k - 2\delta + 1}{2},$$

the midpoint of  $(k - \delta, k + 1 - \delta)$ , one can see that

$$\text{dist}(x_k, \Lambda_\delta) \geq \frac{1}{2}.$$

From (8.7) we conclude that

$$|E_\delta(x_k)| \simeq (1 + x_k)^{2\delta} \text{dist}(x_k, \Lambda_\delta) \gtrsim (1 + x_k)^{2\delta} \simeq k^{2\delta}$$

which goes to infinity as  $k \rightarrow \infty$ . The fact that  $\varphi_\delta$  is unbounded now follows. Complete the proof by transferring things back to the disk by setting via

$$u = I_\delta \circ \omega^{-1}, \quad v = \Theta \circ \omega^{-1}, \quad \varphi = \varphi_\delta \circ \omega^{-1}$$

and applying Lemma 7.3.  $\square$

## 9. MULTIPLIERS AND CLARK MEASURES

Recall our discussion of Clark measures from (2.20). We now exploit these measures to obtain additional information about multipliers. We begin with a simple lemma.

**Lemma 9.1.** *Let  $u, v$  be two inner functions and  $\varphi \in H^2$ . Then  $\varphi \in \mathcal{M}(u, v)$  if and only if there exists a bounded linear operator  $L_\varphi : \mathcal{K}_v \rightarrow \mathcal{K}_u$  satisfying*

$$L_\varphi(k_\lambda^v) = \overline{\varphi(\lambda)} k_\lambda^u, \quad \lambda \in \mathbb{D}.$$

*Proof.* Assume that  $\varphi \in \mathcal{M}(u, v)$ . Then  $M_\varphi$  is a bounded operator from  $\mathcal{K}_u$  into  $\mathcal{K}_v$  and for every  $f \in \mathcal{K}_u$  and  $\lambda \in \mathbb{D}$  we have

$$\langle f, M_\varphi^* k_\lambda^v \rangle = \langle \varphi f, k_\lambda^v \rangle = \varphi(\lambda) f(\lambda) = \varphi(\lambda) \langle f, k_\lambda^u \rangle = \langle f, \overline{\varphi(\lambda)} k_\lambda^u \rangle.$$

Hence  $M_\varphi^* k_\lambda^v = \overline{\varphi(\lambda)} k_\lambda^u$  and thus  $L_\varphi = M_\varphi^*$ . Conversely, assume that the operator  $L_\varphi$ , initially defined on kernel functions by

$$L_\varphi(k_\lambda^v) = \overline{\varphi(\lambda)} k_\lambda^u, \quad \lambda \in \mathbb{D},$$

extends, via linearity, to a bounded operator from  $\mathcal{K}_v$  into  $\mathcal{K}_u$ . For every  $f \in \mathcal{K}_u$ , we have

$$\begin{aligned} \varphi(\lambda) f(\lambda) &= \varphi(\lambda) \langle f, k_\lambda^u \rangle = \langle f, \overline{\varphi(\lambda)} k_\lambda^u \rangle \\ &= \langle f, L_\varphi(k_\lambda^v) \rangle = \langle L_\varphi^*(f), k_\lambda^v \rangle \\ &= (L_\varphi^* f)(\lambda). \end{aligned}$$

Hence  $L_\varphi^* f = \varphi f \in \mathcal{K}_v$  which proves that  $\varphi \in \mathcal{M}(u, v)$ .  $\square$

Here is the rephrasing of the lemma above in terms of the corresponding Clark measures.

**Theorem 9.2.** *Let  $u, v$  be two inner functions and  $\sigma_u, \sigma_v$  be their associated Clark measures. For  $\varphi \in H^2$ , the following are equivalent:*

(i)  $\varphi \in \mathcal{M}(u, v)$ ;

(ii) *there exists a bounded linear operator  $\mathfrak{L}_\varphi : L^2(\sigma_v) \rightarrow L^2(\sigma_u)$  satisfying*

$$\mathfrak{L}_\varphi(k_\lambda) = \overline{\varphi(\lambda)} \frac{1 - \overline{u(\lambda)}}{1 - \overline{v(\lambda)}} k_\lambda, \quad \lambda \in \mathbb{D}.$$

*Proof.* Assume that  $\varphi \in \mathcal{M}(u, v)$ . By Lemma 9.1, the operator  $L_\varphi : \mathcal{K}_v \rightarrow \mathcal{K}_u$  satisfies

$$L_\varphi k_\lambda^v = \overline{\varphi(\lambda)} k_\lambda^u, \quad \lambda \in \mathbb{D}.$$

Define  $\mathfrak{L}_\varphi := V_u^{-1} L_\varphi V_v : L^2(\sigma_v) \rightarrow L^2(\sigma_u)$ , where  $V_u$  was defined earlier in (2.22). Using (2.23) (twice) we obtain

$$\begin{aligned} \mathfrak{L}_\varphi(k_\lambda) &= V_u^{-1} L_\varphi((1 - \overline{v(\lambda)})^{-1} k_\lambda^v) \\ &= (1 - \overline{v(\lambda)})^{-1} \overline{\varphi(\lambda)} V_u^{-1} k_\lambda^u \\ &= \overline{\varphi(\lambda)} \frac{1 - \overline{u(\lambda)}}{1 - \overline{v(\lambda)}} k_\lambda. \end{aligned}$$

Conversely, define the operator  $L_\varphi := V_u \mathfrak{L}_\varphi V_v^{-1}$ . As in the chain of equalities above, we get

$$\begin{aligned} L_\varphi(k_\lambda^v) &= V_u \mathfrak{L}_\varphi V_v^{-1} k_\lambda^v \\ &= V_u \mathfrak{L}_\varphi (1 - \overline{v(\lambda)}) k_\lambda \\ &= \overline{\varphi(\lambda)} V_u ((1 - \overline{u(\lambda)}) k_\lambda) \\ &= \overline{\varphi(\lambda)} k_\lambda^u \end{aligned}$$

To finish, apply Lemma 9.1.  $\square$

**Remark 9.3.** (1) A similar criterion for multipliers of de Branges–Rovnyak spaces  $\mathcal{H}(b)$  appears in [37].

(2) For  $\varphi \in H^2$ , we can use the fact that  $1 - v$  is outer to see that the function

$$\psi := \frac{\varphi(1 - u)}{1 - v}$$

belongs to the Smirnov class  $\mathcal{N}^+$  [17]. According to [48, Proposition 3.1],  $\psi$  can be written uniquely as  $\psi = b/a$  where  $a$  and  $b$  are in the closed unit ball of  $H^\infty$ ,  $a$  is an outer function,  $a(0) > 0$ , and

$$(9.4) \quad |a|^2 + |b|^2 = 1$$

almost everywhere on  $\mathbb{T}$ . It is shown in [48] that if  $T_\psi$  denotes the operator of multiplication by  $\psi$  on the domain

$$\mathcal{D}(T_\psi) = \{f \in H^2 : \psi f \in H^2\},$$

then  $T_\psi$  is a closed and densely defined operator. Moreover,  $\mathcal{D}(T_\psi) = aH^2$  and  $\mathcal{D}(T_\psi^*) = \mathcal{H}(b)$ , the de Branges–Rovnyak space associated with  $b$  [26, 46]. In light of (9.4), it follows from

$$\log(1 - |b|^2) = \log |a|^2 \in L^1$$

(this is a result of Riesz [17, p. 17]) that  $b$  is a non-extreme point of the unit ball of  $H^\infty$  [17, p. 125] and thus  $k_\lambda \in \mathcal{H}(b)$  [46, Ch. IV]. In fact, the linear span of the Cauchy kernels  $\{k_\lambda : \lambda \in \mathbb{D}\}$ , are dense in  $\mathcal{H}(b)$ . From here it is straightforward to show that

$$T_\psi^* k_\lambda = \overline{\psi(\lambda)} k_\lambda, \quad \lambda \in \mathbb{D}.$$

Hence the operator  $T_\psi^*$  and  $\mathfrak{L}_\varphi$  have the same action on the linear span of  $k_\lambda$ , a set which is dense in both  $L^2(\sigma_u)$  and  $\mathcal{H}(b)$ .

**Corollary 9.5.** *Let  $u, v$  be inner with associated Clark measures  $\sigma_u$  and  $\sigma_v$  satisfying  $\sigma_u \ll \sigma_v$ . If  $\varphi = (1 - v)/(1 - u)$  and  $h = d\sigma_u/d\sigma_v$ , the following are equivalent:*

- (i)  $\varphi \in \mathcal{M}(u, v)$
- (ii)  $h \in L^\infty(\sigma_v)$ .

*Proof.* (ii)  $\implies$  (i): Using Theorem 9.2,  $\varphi \in \mathcal{M}(u, v)$  if and only if there exists a bounded linear operator  $\mathfrak{L}_\varphi : L^2(\sigma_v) \rightarrow L^2(\sigma_u)$  such that

$$\mathfrak{L}_\varphi(k_\lambda) = \overline{\varphi(\lambda)} \frac{1 - \overline{u(\lambda)}}{1 - \overline{v(\lambda)}} k_\lambda = k_\lambda, \quad \lambda \in \mathbb{D}.$$

For every  $f \in L^2(\sigma_v)$ , we have

$$\begin{aligned} \int_{\mathbb{T}} |f(\xi)|^2 d\sigma_u(\xi) &= \int_{\mathbb{T}} |f(\xi)|^2 h(\xi) d\sigma_v(\xi) \\ &\leq \|h\|_{L^\infty(\sigma_v)} \|f\|_{L^2(\sigma_v)}^2. \end{aligned}$$

Hence if we define  $\mathfrak{L}_\varphi(f) = f$  for  $f \in L^2(\sigma_v)$ , then the operator  $\mathfrak{L}_\varphi$  is bounded from  $L^2(\sigma_v)$  into  $L^2(\sigma_u)$ , which proves that  $(1 - v)/(1 - u) \in \mathcal{M}(u, v)$ .

(i)  $\implies$  (ii): Again using Theorem 9.2, the map  $\mathfrak{L}_\varphi(k_\lambda) = k_\lambda$  extends linearly to a bounded operator from  $L^2(\sigma_v)$  into  $L^2(\sigma_u)$ . In particular, for any  $f$  in the linear span of  $\{k_\lambda : \lambda \in \mathbb{D}\}$ , we have

$$\int_{\mathbb{T}} |f|^2 h d\sigma_v = \int_{\mathbb{T}} |f|^2 d\sigma_u \lesssim \int_{\mathbb{T}} |f|^2 d\sigma_v.$$

Since the linear span of  $\{k_\lambda : \lambda \in \mathbb{D}\}$  is dense in  $L^2(\sigma_v)$  (use the fact that  $\sigma_v \perp m$  along with the Riesz brothers theorem [29, p. 59]), we see that  $h \in L^\infty(\sigma_v)$ .  $\square$

**Remark 9.6.** It was shown in [46] that if  $\sigma_u \ll \sigma_v$  and  $h := d\sigma_u/d\sigma_v$ , then  $h \in L^2(\sigma_v)$  if and only if  $(1 - v)/(1 - u) \in H^2$ .

**Example 9.7.** If  $v$  is the atomic inner function

$$v(z) = \exp\left(-\frac{1+z}{1-z}\right),$$

then it is easy to see that  $v(z) = 1$  if and only if

$$z = z_n = \frac{2i\pi n - 1}{2i\pi n + 1}, \quad n \in \mathbb{Z}.$$

Hence the Clark measure  $\sigma_v$  is discrete and given by

$$\sigma_v = \sum_{n \in \mathbb{Z}} c_n \delta_{z_n},$$

where

$$c_n = \frac{1}{|v'(z_n)|} = \frac{2}{4\pi^2 n^2 + 1}.$$

Now pick  $c'_n$  satisfying  $0 \leq c'_n \leq M c_n$  for some  $M \geq 1$  and define

$$\mu' = \sum_{n \geq 1} c'_n \delta_{z_n}.$$

See [27, Ch. 11] for the details on this. In other words, we have  $d\mu' = h d\sigma_v$ , where  $0 \leq h \leq M$ . By (2.21) there is a unique inner function  $u$  such that its associated Clark measure is precisely  $\mu'$ . Corollary 9.5 now says that

$$\frac{1-v}{1-u} \in \mathcal{M}(u, v).$$

This construction can be done more generally starting from any finite measure  $\sum_{n \geq 1} c_n \delta_{z_n}$  on  $\mathbb{T}$  and its associated inner function  $v$ . A version of the construction above appears in [28].

When  $\sigma_u \ll \sigma_v$  and  $h = d\sigma_u/d\sigma_v \in L^\infty(\sigma_v)$ , is it possible to describe all the multipliers from  $\mathcal{K}_u$  into  $\mathcal{K}_v$ ? Could it be that

$$\mathcal{M}(u, v) = \mathbb{C} \frac{1-v}{1-u}?$$

## 10. REVISITING CROFOOT'S CHARACTERIZATION

To state our characterization of the onto multipliers, we need a definition and a preliminary result. For a function  $w : \mathbb{T} \rightarrow [0, \infty)$ , we say that the measure  $w dm$  is a *Carleson measure* for  $\mathcal{K}_u$  when

$$\int_{\mathbb{T}} |f|^2 w dm \lesssim \int_{\mathbb{T}} |f|^2 dm, \quad f \in \mathcal{K}_u$$

and a *reverse Carleson measure* for  $\mathcal{K}_u$  when

$$\int_{\mathbb{T}} |f|^2 w dm \gtrsim \int_{\mathbb{T}} |f|^2 dm, \quad f \in \mathcal{K}_u.$$

We point the reader to the survey paper [25] on reverse Carleson measures on various spaces of analytic functions and the paper [5] specifically for reverse Carleson measures on model spaces.

If  $\varphi\mathcal{K}_u = \mathcal{K}_v$  (i.e.,  $\varphi$  is an onto multiplier), Remark 3.13 says that  $\varphi$  and  $\frac{1}{\varphi}$  belong to  $H^2$  and thus  $\varphi$  is an outer function. Moreover,  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$  and  $|\varphi|^{-2} dm$  is a Carleson measure for  $\mathcal{K}_v$ .

We need the following result from [52]: Let  $w : \mathbb{T} \rightarrow [0, \infty)$  be bounded and let  $u$  be inner. Then

$$\int_{\mathbb{T}} |f|^2 w dm \asymp \int_{\mathbb{T}} |f|^2 dm, \quad f \in \mathcal{K}_u$$

if and only if

$$\inf\{\widehat{w}(\lambda) + |u(\lambda)| : \lambda \in \mathbb{D}\} > 0,$$

where

$$\widehat{w}(\lambda) = \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\xi - \lambda|^2} w(\xi) dm(\xi)$$

is the Poisson extension of  $w$  to  $\mathbb{D}$ . Since  $w$  is bounded, then  $w dm$  is automatically a Carleson measure and thus the above infimum condition tests whether or not  $w dm$  is a reverse Carleson measure for  $\mathcal{K}_u$ .

Here is our characterization of the onto multipliers.

**Theorem 10.1.** *For inner functions  $u, v$  and  $\varphi \in H^2$ , the following are equivalent:*

- (i)  $\varphi\mathcal{K}_u = \mathcal{K}_v$ ;
- (ii) *The following three conditions hold:*
  - (a)  $\varphi S^* u \in \mathcal{K}_v$ ;
  - (b)  $\frac{1}{\varphi} \in H^2$  and  $\frac{1}{\varphi} S^* v \in \mathcal{K}_u$ ;
  - (c)  $|\varphi|^2 dm$  is both a Carleson and a reverse Carleson measure for  $\mathcal{K}_u$ .

*Proof.* To prove (i)  $\implies$  (ii) we note from our preliminary remarks that conditions (a) and (b) are satisfied. We also see that condition (i) implies that  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$  and  $|\varphi|^{-2} dm$  is a Carleson measure for  $\mathcal{K}_v$ . In particular, this means there is a constant  $c > 0$  such that

$$\int_{\mathbb{T}} |g|^2 |\varphi|^{-2} dm \leq c \int_{\mathbb{T}} |g|^2 dm, \quad g \in \mathcal{K}_v.$$

For  $f \in \mathcal{K}_u$  we have  $g = \varphi f \in \mathcal{K}_v$  and the preceding inequality yields

$$\int_{\mathbb{T}} |f|^2 dm \leq c \int_{\mathbb{T}} |f|^2 |\varphi|^2 dm$$

and thus  $|\varphi|^2 dm$  is both a Carleson and a reverse Carleson measure for  $\mathcal{K}_u$ . Thus condition (c) is satisfied.

To prove (ii)  $\implies$  (i) we observe that the conditions  $\varphi S^*u \in \mathcal{K}_v$  and  $|\varphi|^2 dm$  is a Carleson measure for  $\mathcal{K}_u$  imply that  $\varphi \mathcal{K}_u \subseteq \mathcal{K}_v$  (Theorem 4.1). Now since  $|\varphi|^2 dm$  is a reverse Carleson measure for  $\mathcal{K}_u$ , we have

$$\int_{\mathbb{T}} |f|^2 dm \lesssim \int_{\mathbb{T}} |f|^2 |\varphi|^2 dm, \quad f \in \mathcal{K}_u.$$

As in the proof of Theorem 4.1 we see that since  $\frac{1}{\varphi} S^*v \in \mathcal{K}_v$  we have  $\frac{1}{\varphi} g \in \mathcal{K}_u$  for all  $g \in \mathcal{S} := \text{span}\{S^{*n}v : n \geq 1\}$ . For any  $g \in \mathcal{S}$  we use the previous inequality to see that

$$\int_{\mathbb{T}} |g|^2 |\varphi|^{-2} dm \lesssim \int_{\mathbb{T}} |g|^2 dm$$

which means, since  $\mathcal{S}$  is dense in  $\mathcal{K}_v$ , that  $\frac{1}{\varphi} \mathcal{K}_v \in \mathcal{K}_u$ .  $\square$

Combining this result and our discussion above, we have the following characterization of the *bounded* onto multipliers.

**Corollary 10.2.** *For inner functions  $u, v$  and  $\varphi \in H^\infty$ , the following are equivalent:*

$$(i) \quad \varphi \mathcal{K}_u = \mathcal{K}_v;$$

$$(ii) \quad \varphi S^*u \in \mathcal{K}_v, \quad \frac{1}{\varphi} \in H^2, \quad \frac{1}{\varphi} S^*v \in \mathcal{K}_u, \quad \text{and}$$

$$\inf\{\widehat{|\varphi|^2}(\lambda) + |u(\lambda)| : \lambda \in \mathbb{D}\} > 0;$$

**Remark 10.3.** For  $\varphi \in H^\infty$  notice that  $|\varphi(\lambda)| \leq \widehat{|\varphi|^2}(\lambda)$  for all  $\lambda \in \mathbb{D}$ . Thus if  $\varphi S^*u \in \mathcal{K}_v$ ,  $\frac{1}{\varphi} \in H^2$ ,  $\frac{1}{\varphi} S^*v \in \mathcal{K}_u$ , and

$$\inf\{|\varphi(\lambda)|^2 + |u(\lambda)| : \lambda \in \mathbb{D}\} > 0,$$

then  $\varphi \mathcal{K}_u = \mathcal{K}_v$ .

## 11. GENERALIZATIONS

When  $p \in (1, \infty)$  there is an  $L^p$  version of the model spaces  $\mathcal{K}_u$ , namely, via boundary values as in (2.7),

$$\mathcal{K}_u^p := H^p \cap u(\overline{\zeta H^p}),$$

where

$$H^p := \left\{ f \in L^p : \int_{\mathbb{T}} f(\xi) \bar{\xi}^n dm(\xi) = 0 \quad \forall n < 0 \right\}.$$



From here one can examine the multipliers

$$\mathcal{M}_p(u, v) := \{\varphi \in \mathcal{O}(\mathbb{D}) : \varphi \mathcal{K}_u^p \subseteq \mathcal{K}_v^p\}.$$

It is still the case that  $\mathcal{M}_p(u, v) \subseteq H^p$  and our main characterization (Theorem 4.1) holds for  $\mathcal{M}_p(u, v)$  with the appropriate change that  $|\varphi|^p dm$  is a Carleson measure for  $\mathcal{K}_u^p$ . Theorem 3.14 is also valid in this more general setting but the Crofoot-transform is no longer isometric. However, certain  $L^p$  results concerning bounded multipliers are more delicate since the norm of a reproducing kernel no longer has the nice form except when  $u$  and  $v$  satisfy the connected level set condition (see [3, Corollary to Theorem 1.3]). In particular, equation (3.9) and Corollary 5.2 still work under the connected level set condition.

It would be worth exploring the dependence of  $\mathcal{M}_p(u, v)$  on the parameter  $p$ . For example, Dyakonov [21] showed that for certain  $\varphi \in L^\infty$ ,

$$\text{Ker}_p T_\varphi = \{f \in H^p : P_+(\varphi f) = 0\}$$

can depend quite dramatically on  $p$ . Indeed,  $\dim \text{Ker}_p T_\varphi$  can change with  $p$ . Does the same “jump in dimension” phenomenon take place with the multiplier space  $\mathcal{M}_p(u, v)$ ?

We mention that in the context of multipliers from  $\mathcal{K}_u^p$  to  $\mathcal{K}_v^q$ , where  $u$  divides  $v$ , Dyakonov characterizes the corresponding Carleson measures (see for instance [22, Theorem A]).

Recall the de Branges-Rovnyak spaces  $\mathcal{H}(b)$  mentioned earlier. These are close cousins to the model spaces, where, in fact,  $\mathcal{H}(b) = \mathcal{K}_b$  when  $b$  is inner. One can ask, as was done in [28], about the multipliers  $\mathcal{M}(\mathcal{H}(b_1), \mathcal{H}(b_2))$ . The paper [28] also relates other multiplier problems (e.g., between two Herglotz spaces) to pre-order problems in operator theory. See also [14] where some general facts are stated in the setting of de Branges-Rovnyak spaces.

Finally, one can ask, what are necessary and sufficient conditions on  $u$  and  $v$  such that  $\mathcal{M}(u, v) \neq \{0\}$ ? This non-triviality of  $\mathcal{M}(u, v)$  goes beyond the non-triviality of the kernel of a Toeplitz operator and appears in [28] in the context of pre-orders of partial isometries. This current paper contains several results along these lines (some necessary, some sufficient) but a tractable necessary and sufficient condition remains unknown.

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